FALL 2022: MATH 996 DAILY UPDATE

Wednesday, December 7. A. Pascoe and E. Salcido presented the paper *Homological algebra on a complete intersection with an application to group representations*, by D. Eisenbud.

Monday, December 5. W. King and S. Suresh presented the paper *Form rings and regular sequences*, by P. Valabrega and G. Valla.

Friday, December 2. S. Das and R. Dutta presented the paper *Every algebraic set in n-space is the intersection of n hypersurfaces*, by D. Eisenbud and E.G. Evans.

Wednesday, November 30. We began class by stating and proving the following results.

Proposition. Let R be a Noetherian ring and M an R-module. The following statements are equivalent:

(i) inj.dim.$(M) \leq n$.
(ii) $\text{Ext}_R^{n+1}(R/J, M) = 0$, for all ideals $J \subseteq R$.
(iii) $\text{Ext}_R^{n+1}(R/P, M) = 0$, for all prime ideals $P \subseteq R$.

Lemma. Let $(R, \mathfrak{m}, k)$ be a local ring, M a finitely generated R-module and $P \subsetneq \mathfrak{m}$ a prime ideal. If $\text{Ext}_R^{i+1}(R/Q, M) = 0$ for all Q properly containing P, then $\text{Ext}_R^i(R/P, M) = 0$.

Both proofs used long exact Ext sequences to prove the required vanishing statements. The proof of the crucial implication (ii) implies (i) in the Proposition was by induction on n, exploiting the fact that Baer's Criterion can be re-stated as saying an R-module Q is injective if and only if $\text{Ext}_R^1(R/J, Q) = 0$, for all ideals $J \subseteq R$. With these result, we were able to prove the following theorem.

Theorem. Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated module. Then inj.dim.$(M) = n$ if and only if $\text{Ext}_R^n(k, M) \neq 0$ and $\text{Ext}^i(k, M) = 0$, for all $i > n$.

In order to see that a Noetherian local ring R is zero-dimensional, if it is an injective R-module, we needed the following standard result.

Proposition. Let R be a Noetherian ring and Q an injective R-module. Let $J \subseteq R$ be an ideal. Then $(0 :_Q J^\infty)$ is an injective R-module.

The proof used the Artin-Rees lemma, roughly as follows. Suppose $I \subseteq R$ is an ideal and $f : I \to (0 :_Q J^\infty)$ is an R-module homomorphism. One needs $\rho : R \to (0 :_Q J^\infty)$ such that $f(i) = \rho(i)$ for all $i \in I$. There is a map $\rho_0 : R \to Q$ such that $f(i) = \rho_0(i)$, for all $i \in I$. One uses Artin-Rees to show that $J^n\rho_0(i) \cap f(I) = 0$, for $n >> 0$. Fixing such an n, this then enables one to define $\tilde{f} : J^n + I \to Q$, extending f, in such a way that there exists $\rho : R \to Q$ with $\tilde{f}(t) = \rho(t)$, for all $t \in J^n + I$, and $\rho(1) \in (0 :_Q J^\infty)$, which gives the result. The

The indecomposability of a local ring R as an R-module, gave the following corollary to the previous proposition.

Corollary. Let $(R, \mathfrak{m})$ be a local ring that is an injective R-module. Then R has Krull dimension zero.

The point of the proof of the corollary was that R must have depth zero, and hence $(0 : \mathfrak{m}^\infty)$ is an injective submodule of R and thus, a summand of R and therefore must equal R. This result serves as the base case for the if direction of the final theorem for our course.

Theorem. Let $(R, \mathfrak{m})$ be a local ring of dimension d. Then R is Gorenstein if and only if R inj.dim.$(R) < \infty$, in which case inj.dim.$(R) = d$.

The proof was an easy induction argument using Rees's Ext theorem, the theorem above characterizing injective dimension, and the fact that if $x \in R$ is a non-zerodivisor, R is Gorenstein (respectively, has finite injective dimension as an R-module) if and only if R/xR is Gorenstein (respectively, has finite injective dimension as an R/xR-module).

We ended class by noting (but not proving) the following relevant facts: (i) If R is local ring and M is a finitely generated R-module with finite injective dimension, then, in fact, $\text{inj.dim.}(M) = \text{depth}(R)$ and (ii) If $(R, \mathfrak{m})$ is a Gorenstein local ring, and M a finitely generated R-module, then $\text{inj.dim.}(M) < \infty$ if and only if $\text{proj.dim.}(M) < \infty$, a theorem due to W. Vasconcelos.

Monday, November 28. We began class by noting (but not formally proving) that if $I \subseteq R$ is an ideal and M an R-module, then $\text{Hom}_R(R/I, M) \cong (0 :_M I) =: \{x \in M \mid I \cdot x = 0\}$. When $(R, \mathfrak{m}, k)$ is a local ring and M is an R-module, we also defined the *socle* of M as $\text{soc}(M) := \text{Hom}(k, M)$ and noted that when M is a finitely generated R-module, $\text{soc}(M)$ is a finite dimensional vector space over k. We then stated and proved:

Theorem. Let $(R, \mathfrak{m}, k)$ be a zero-dimensional local ring. The following are equivalent:

(i) $\dim_k(\text{soc}(R)) = 1$.
(ii) $(0 : (0 : I)) = I$, for all ideals $I \subseteq R$.
(iii) R is an injective R-module (sometimes said to be *self-injective*).

The zero-dimensional local ring $(R, \mathfrak{m}, k)$ was then defined to be *Gorenstein* if any, and hence all, of the conditions in the theorem above hold. As a corollary, we noted that a zero-dimensional local ring is Gorenstein if and only if (0) is an irreducible ideal, i.e., (0) cannot be written as the intersection of two non-zero ideals. This was followed by stating (but not proving) the following *change of rings* theorem due to D. Rees: Let R be a Noetherian ring, and M, N finitely generated R-modules. Suppose $0 \neq x \in R$ annihilates N and is a non-zerodivisor on M. Then, $\text{Ext}_R^{i+1}(N, M) \cong \text{Ext}_{R_1}^i(N, M/xM)$, for all $i \geq 0$ and $R_1 := R/xR$.

We were then able to state the following:

Proposition-Definition. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. The following are equivalent:

(i) R is Cohen-Macaulay and $\dim_k \text{Ext}^d(k, R) = 1$.
(ii) R is Cohen-Macaulay and for any $\underline{x} \subseteq R$, a system of parameters in R, $\dim_d \text{soc} R/\langle \underline{x} \rangle = 1$.
(iii) R is Cohen-Macaulay and for any $\underline{x} \subseteq R$, a system of parameters in R, $I = (\underline{x} : (\underline{x} : I))$, for all ideals I containing $\underline{x}$.
(iv) R is Cohen-Macaulay and for any $\underline{x} \subseteq R$, a system of parameters in R, $\langle \underline{x} \rangle$ is an irreducible ideal.

R is said to be Gorenstein if any, and hence all, of the conditions above hold. We noted that the equivalence of the conditions in the Proposition follows immediately by applying Rees's theorem and the theorem characterizing zero-dimensional Gorenstein local rings.

We ended class by noting that our final lecture will be devoted to proving that a local ring R is Gorenstein if and only if R has finite injective dimension as an R-module.

Monday, November 21. We began with an overview of injective modules and the Ext functor with an eye towards discussing the definitions and basic properties of Gorenstein rings. After defining injective modules, we gave two examples, the first showing that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module and the second showing that $\mathbb{Z}$ is *not* an injective $\mathbb{Z}$-module. We then stated and proved:

Baer's Criterion. An R-module Q is injective if and only if for all ideals $I \subseteq R$ and R-module homomorphisms $f : I \to Q$, there exists $\rho : R \to Q$ such that $i\rho = f$, where $i : I \to R$ is the inclusion map.

We then stated but did not prove the following proposition:

Proposition. For an R-module Q, the following are equivalent:

(i) Q is an injective R-module.
(ii) Whenever $Q \subseteq M$, for an R-module M, there exists an R-module K such that $M = Q \oplus K$.
(iii) For every exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, the sequence

$$0 \to \text{Hom}(C, Q) \xrightarrow{\beta^*} \text{Hom}(B, Q) \xrightarrow{\alpha^*} \text{Hom}(A, Q) \to 0$$

is exact.

The proposition above was followed by outlining the steps needed to prove the following fundamental fact: Every R-module is isomorphic to a submodule of an injective module. This then enabled us to note that every R-module has an *injective resolution.* For R-modules M and N we defined $\mathrm{Ext}^i_R(M,N)$ to be either the ith cohomology module in the complex obtained by taking a projective resolution of M and applying $\mathrm{Hom}(-,N)$ or the ith cohomology module in the complex obtained by applying $\mathrm{Hom}(M,-)$ to an injective resolution of N. We did not prove the standard facts that these definitions are well defined and agree. We then noted, but did not prove, the following facts:

(a) N has injective dimension equal to n if and only if $\mathrm{Ext}^i_R(L,N)=0$, for all $i>n$, all L and $\mathrm{Ext}^n_R(L_0,N)\neq 0$, for some L_0.
(b) M has projective dimension equal to n if and only if $\mathrm{Ext}^i_R(M,L)=0$, for all $i>n$, all L and $\mathrm{Ext}^n_R(M,L_0)\neq 0$, for some L_0.

We noted, but did not write down, the long exact sequences involving Ext needed to prove properties (a) and (b) above, namely, if $0\to A\to B\to C\to 0$ is exact, then given M,N there exist long exact sequences

$$0\to \mathrm{Hom}_R(C,N)\to \mathrm{Hom}_R(B,N)\to \mathrm{Hom}_R(A,N)\to 0\to \mathrm{Ext}^1_R(C,N)\to \mathrm{Ext}^1_R(B,N)\to \mathrm{Ext}^1_R(A,N)\to\cdots$$

$$0\to \mathrm{Hom}_R(M,A)\to \mathrm{Hom}_R(M,B)\to \mathrm{Hom}_R(M,C)\to 0\to \mathrm{Ext}^1_R(M,A)\to \mathrm{Ext}^1_R(M,B)\to \mathrm{Ext}^1_R(M,C)\to\cdots$$

We ended class by stating, but not proving, the following theorem which plays an important role in establishing properties of Gorenstein rings:

Theorem. Let $(R,\mathfrak{m})$ be a local ring with residue field k. A finitely generated R-module M has injective dimension n if and only if $\mathrm{Ext}^n_R(k,M)\neq 0$ and $\mathrm{Ext}^i_R(k,M)=0$, for all $i>n$.

Friday, November 18. We began class by proving the following proposition:

Proposition. Let $(R,\mathfrak{m})$ be a local ring and M a finitely generated R-module of dimension d. Suppose $\underline{x}=x_1,x_2,\ldots,x_d$ and $\underline{x}'=x_1',x_2,\ldots,x_d$ are system of parameters on M. Then: $x_1x_1',x_2,\ldots,x_d$ is a system of parameters on M and

$$\chi(x_1x_1',x_2,\ldots,x_d;M)=\chi(\underline{x};M)+\chi(\underline{x}';M).$$

The key to the proof was using the complex $\mathcal{E}:0\to R^2\xrightarrow{A}R^2\to 0$, where $A=\begin{pmatrix}x_1 & 1\\ 0 & x_1'\end{pmatrix}$. We then constructed two exact sequences of complexes:

(a) $0\to\mathcal{K}(x_1)\to\mathcal{E}\to\mathcal{K}(x_1')\to 0$
(b) $0\to\mathcal{K}(x_1x_1')\to\mathcal{E}\to\mathcal{K}(1)\to 0$

which upon tensoring with $\mathcal{K}(x_2,\ldots,x_d;M)$ led to

$$\begin{aligned}\chi(\mathcal{E}\otimes\mathcal{K}(x_2,\ldots,x_d;M)) &= \chi(\underline{x};M)+\chi(\underline{x}';M)\\ &= \mathcal{K}(x_1x_1',x_2,\ldots,x_d;M)),\end{aligned}$$

which gives the result.

We then stated and gave a proof of:

Lech's Lemma. Let $(R,\mathfrak{m})$ be a local ring, M a finitely generated R module and $\underline{x}=x_1,\ldots,x_d$ a system of parameters on M (so that M has dimension d). Then,

$$e(\underline{x};M)=\lim_{n\to\infty}\frac{1}{n^d}\cdot\lambda(M/(x_1^n,\ldots,x_d^n)M).$$

The proof followed by noting that the previous proposition implies that $\chi(x_1^n,\ldots,x_d^n;M)=n^d\chi(\underline{x};M)$ and applying the fact that for $1\leq i\leq d$, the lengths of the Kozsul homology modules $\mathrm{H}_i(x_1^n,\ldots,x_d^n;M)$ are bounded by a constant times n^{d-1}, for $n>>0$.

We then had a brief discussion concerning Serre's intersection multiplicity: If $(A,\mathfrak{m})$ is a local ring of dimension d, M,N finite A-modules such that $\lambda(M\otimes N)<\infty$, and one of M or N has finite projective dimension, then $\chi(M,N):=\Sigma_{i=1}^d(-1)^i\lambda(\mathrm{Tor}_i^R(M,N))$. The purpose of the discussion was to indicate, but not prove, how Serre derived results involving $\chi(M,N)$ when A is a regular local ring (containing a field) by an algebraic version of the geometric technique of *reduction to the diagonal*. The basic idea is that if A is say, the local ring obtained by localizing the polynomial ring $k[x_1, \ldots, x_d]$ at its homogeneous maximal ideal, then $\chi(M, N) = \chi_B(\underline{D}, M \otimes_k M)$, where $B = A \otimes_k A$ (the polynomial ring in $2d$ variables localized at its homogeneous maximal ideal) and $D \subseteq B$ is the ideal generated by the regular sequence $x_1 - y_1, \ldots, x_d - y_d$, so that $\chi(M, N)$ become the type of multiplicity we have been looking at in our recent lectures. We also noted that for a general regular local ring containing a field, one may pass to the completion, so that $A = k[[x_1, \ldots, x_d]]$ is a power series ring in d variables. But then $A \otimes_k A$ is not a power series ring in $2d$ variables. We finished the discussion by noting that to overcome this problem, one needs to construct *completed tensor products*, and that with this construction, $A \hat{\otimes}_k A$ is a power series ring in $2d$ variables, and after several techincal preliminaries, the technique of reduction to the diagonal goes through in this case as well.

Wednesday, November 16. We spent most of the class proving the following proposition:

Proposition. Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module and $\underline{x} := x_1, \ldots, x_k \in R$ a sequence of elements such that $\lambda(M/\underline{x}M) < \infty$. Suppose $d := \dim(M)$. Then:

(i) There exists $c > 0$ such that $\lambda(\mathrm{H}_i(x_1^n, \ldots, x_k^n; M)) \leq cn^d$, for all i and all $n >> 0$.
(ii) If $k = d$, there exists $c > 0$ such that $\lambda(\mathrm{H}_i(x_1^n, \ldots, x_d^n; M)) \leq cn^{d-1}$, for $1 \leq i \leq d$ and all $n >> 0$.

The proof proceeded by induction on d, starting with $d = 0$ in the first case and $d = 1$ in the second case. The proof used subadditivity of lengths in long exact sequences of Koszul homology, and the following two facts:

(i) $\lambda(\mathrm{H}_i(x_1^n, \ldots, x_k^n; M)) = \lambda(\mathrm{H}_i(x_2^n, \ldots, x_k^n; M/x_1^n M))$ when x_1 is a non-zerodivisor on M
(ii) $\lambda(\mathrm{H}_i(x_2^n, \ldots, x_k^n; M/x_1^n M)) \leq n \cdot \lambda(\mathrm{H}_i(x_2^n, \ldots, x_k^n; M/x_1^n M))$.

We then stated the following proposition.

Proposition. Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module of dimension d. Suppose $\underline{x} = x_1, x_2, \ldots, x_d$ and $\underline{x}' = x_1', x_2, \ldots, x_d$ are system of parameters on M. Then: $x_1x_1', x_2, \ldots, x_d$ is a system of parameters on M and

$$\chi(x_1x_1', x_2, \ldots, x_d; M) = \chi(\underline{x}; M) + \chi(\underline{x}'; M).$$

In preparation for the proof of the proposition we first defined the tensor product of complexes, and then noted that if $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{F} \to 0$ is a short exact sequence of complexes such that the modules in $\mathcal{F}$ are free modules, then, for any complex $\mathcal{C}$, $0 \to \mathcal{D} \otimes \mathcal{C} \to \mathcal{D} \otimes \mathcal{C} \to \mathcal{F} \otimes \mathcal{C} \to 0$ is an exact sequence of complexes.

Monday, November 14. We continued our discussion of the mapping cone construction $\mathcal{C} \otimes \mathcal{K}(x)$ with the following proposition:

Proposition. Let $\mathcal{C}$ be a complex of R-modules and $x \in R$.

(i) If x is such that the map of complexes $\mathcal{C} \xrightarrow{\cdot x} \mathcal{C}$ is injective, then for all n, $\mathrm{H}_n(\mathcal{C} \otimes \mathcal{K}(x)) \cong \mathrm{H}_n(\mathcal{C}/x\mathcal{C})$.
(ii) $\mathrm{H}_n(\mathcal{C} \otimes \mathcal{K}(0)) \cong \mathrm{H}_n(\mathcal{C}) \oplus \mathrm{H}_{n-1}(\mathcal{C})$, for all n.

We noted that item (ii) above shows that Koszul homology, and thus, the intersection multiplicity defined by a sequence $\underline{x}$, depends upon the sequence $\underline{x}$, and not the ideal generated by $\underline{x}$. We followed this by indicating how to show that if $\underline{x} = x_1, \ldots, x_d$ and $\underline{x}' = x_1, \ldots, x_{d-1}$ are sequences of elements from R, then for any finitely generated module M, we have isomorphic complexes $\mathcal{K}(\underline{x}; M) \cong \mathcal{K}(\underline{x}'; M) \otimes \mathcal{K}(x_d)$. The results on our mapping cone construction shows that for an R-module M, there is a short exact sequence of complexes $0 \to \mathcal{K}(\underline{x}'; M) \to \mathcal{K}(\underline{x}; M) \to \mathcal{K}(\underline{x}'; M)(-1) \to 0$, such that in the associated long exact sequence in Koszul homology, the connecting homomorphism is multiplication by $\pm x_d$. We then noted (with very brief discussions) the following facts:

(i) If $\underline{x}$ is a regular sequence on M, then the complex $\mathcal{K}(\underline{x}; M)$ is acyclic, and thus, $\mathcal{K}(\underline{x})$ provides a free resolution of $R/\langle \underline{x} \rangle$.
(ii) If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then there is a short exact sequence of complexes $0 \to \mathcal{K}(\underline{x}; A) \to \mathcal{K}(\underline{x}; B) \to \mathcal{K}(\underline{x}; C) \to 0$ and hence a corresponding long exact sequence in homology.
(iii) If $\underline{x}^\sigma$ denotes the permutation of the elements $\underline{x}$ given by $\sigma \in S_d$, then $\mathcal{K}(\underline{x}^\sigma; M) \cong \mathcal{K}(\underline{x}; M)$.

(iv) If $0 \to A \to B \to C \to 0$ is an exact sequence of modules such that $B/\underline{x}B$ has finite length, then $\chi(\underline{x}; B) = \chi(\underline{x}; A) + \chi(\underline{x}; C)$.

We noted that (iv) follows since the alternating sum of the lengths of the homology modules in the associated long exact sequence in Koszul homology is zero, and thus bringing the alternating sum of the lengths of the homology modules $\mathrm{H}_i(\underline{x}; B)$ to one side of this equation gives (iv).

Friday, November 11. We began class by proving that the complex $\mathcal{K}(n)$ from the previous lecture is exact for $n >> 0$. The proof relied on the Artin-Rees lemma and the fact that multiplication by each x_i on $\mathcal{K}$ is null homotopic. We then noted that the main theorem from the previous lecture has an analogue for finitely generated modules, and essentially the same proof carries over for modules. We then defined the *intersection multiplicity* for a sequence of elements $\underline{x} = x_1, \dots, x_d \in R$ and a finite R-module M satisfying $\lambda(M/\underline{x}M) < \infty$ to be the *Euler characteristic* of the Koszul complex, that is, $\chi(\underline{x}; M) = \Sigma_{i=1}^{d}(-1)^i\lambda(\mathrm{H}_i(\underline{x}; M))$, where $\mathrm{H}_i(\underline{x}; M)$ is the ith Koszul homology in the Koszul complex $\mathcal{K}(\underline{x}; M)$ on M.

As a means for deriving facts about intersection multiplicity via properties of the Koszul complex, we made the following definition: Given a complex $\mathcal{C} : \cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$ of R-modules and $x \in R$, $\mathcal{C}\otimes\mathcal{K}(x)$ is the complex whose nth module is $C_n \oplus C_{n-1}$ and whose maps $\delta : C_n \oplus C_{n-1} \to C_{n-1} \oplus C_{n-1}$ are given by $\delta(c_n, c_{n-1}) = (\partial(c_n) + (-1)^{n-1}x \cdot c_{n-1}, \partial(c_{n-1}))$. We finished class by showing that there is a short exact sequence of complexes $0 \to \mathcal{C} \to \mathcal{C} \otimes \mathcal{K}(x) \to \mathcal{C}(-1) \to 0$ whose connecting homomorphisms are multiplication by $\pm x$.

Wednesday, November 9. We continued along a path leading to the proof of:

Theorem. Let $(R, \mathfrak{m})$ be a local ring of dimension d, $\underline{x}$ a system of parameters, $\mathcal{K}$ the Koszul complex on $\underline{x}$ and $\mathrm{H}_i(\underline{x})$ the ith homology module of $\mathcal{K}$. Then, for $I := \langle \underline{x} \rangle$, $e(I) = \Sigma_{i=0}^{d}(-1)^i\lambda(\mathrm{H}_i(\underline{x}))$.

We first presented two more preliminary results:

(iii) If $(R, \mathfrak{m})$ is a local ring of dimension d and $I \subseteq R$ is an $\mathfrak{m}$-primary ideal, with Hilbert-Samuel polynomial $P_I(n)$, then for any $1 \le r \le d$, $\Delta^r(P_I(n)) = \Sigma_{j=0}^{r}(-1)^j\binom{d}{j}P_I(n-j)$.

(iv) If $\mathcal{K}$ is the Koszul complex on a set of elements in $x_1, \dots, x_d$, then for each $1 \le i \le d$ the map of complexes $\mathcal{K} \xrightarrow{\cdot x_i} \mathcal{K}$ is null homotopic.

We then presented the proof of the theorem. The proof required the consideration of the subcomplex $\mathcal{K}(n)$ of $\mathcal{K}$ for $n \ge d$ defined as follows:

$$\mathcal{K}(n) := 0 \to I^{n-d}K_d \longrightarrow I^{n-d+1}K_{d-1} \longrightarrow \cdots \longrightarrow I^{-1}K_1 \longrightarrow I^nK_0 \to 0,$$

The crucial point being that for $n >> 0$, $\mathcal{K}(n)$ is exact. From this it followed that complexes $\mathcal{K}$ and $\mathcal{K}/\mathcal{K}(n)$ have isomorphic homology modules and thus

$$\begin{aligned}
\Sigma_{i=0}^{d}(-1)^i\lambda(\mathrm{H}_i(\underline{x})) &= \Sigma_{i=0}^{d}(-1)^i\lambda(\mathrm{H}_i(\mathcal{K}/\mathcal{K}(n)) \\
&= \Sigma_{i=0}^{d}(-1)^i\lambda((\mathcal{K}/\mathcal{K}(n))_i), \quad \text{by preliminary result (i)} \\
&= \Sigma_{i=0}^{d}(-1)^i\binom{d}{i}\lambda(R/I^{n-i}) \\
&= \Delta^d(P_i(n), \quad \text{by preliminary result (iii)} \\
&= e(I) \quad \text{by preliminary result (ii).}
\end{aligned}$$

The proof of the crucial point mentioned above was postponed until the next lecture.

Monday, November 7. We began class by proving the remaining part of the theorem from the previous lecture, namely, that if R is a local ring and $I \subseteq R$ is an ideal generated by a system of parameters, then R is Cohen-Macaulay if $e(I) = \lambda(R/I)$. This then led to a **very informal** discussion about intersection multiplicities and why, for example, a single length alone does not always give the required multiplicity, just as $e(I)$ need not equal $\lambda(R/I)$ in the absence of the Cohen-Macaulay condition. We noted that the theorem below can be thought of as a precursor to the intersection multiplicity defined by Serre, a special case of which is the following: If $I, J \subseteq R$ are ideals in the local ring R such that $R(I + J)$ has finite length, then $\chi(R/I, R/J) := \Sigma_{i=0}^{d}(-1)^i\lambda(\mathrm{Tor}_i(R/I, R/J))$ is the intersection multiplicity of R/I and R/J. We then stated the:

Theorem. Let $(R, \mathfrak{m})$ be a local ring of dimension d, $\underline{x}$ a system of parameters, $\mathcal{K}$ the Koszul complex on $\underline{x}$ and $\mathrm{H}_i(\underline{x})$ the ith homology module of $\mathcal{K}$. Then, for $I := \langle \underline{x} \rangle$, $e(I) = \Sigma_{i=0}^{d}(-1)^i\lambda(\mathrm{H}_i(\underline{x}))$.

We followed the statement of the theorem by proving the following preliminary results:

(i) Let $\mathcal{C}: \quad 0 \to C_r \xrightarrow{\phi_r} C_{r-1} \xrightarrow{\phi_{r-1}} \cdots \longrightarrow C_1 \xrightarrow{\phi_1} C_0 \to 0$ be a complex of finite length R-modules. Then $\Sigma_{i=0}^{r}(-1)^r\lambda(C_i) = \Sigma_{i=0}^{r}(-1)^r\lambda(\mathrm{H}_i(\mathcal{C}))$, where $\mathrm{H}_i(\mathcal{C})$ is the ith homology module of $\mathcal{C}$.

(ii) If $P_I(n)$ is the Hilbert polynomial of the $\mathfrak{m}$-primary ideal I, then $\Delta^d(P_I(n)) = e(I)$, where for any numerical function $f(n)$, $\Delta(f(n)) := f(n) - f(n-1)$ and $\Delta^r(f(n)) := \Delta(\Delta^{r-1}(f(n)))$, for $r > 1$.

Friday, November 4. We began class by finishing the proof of the Superficial Element Lemma from the previous lecture. We then observed that the proof of the main theorem from the previous lecture shows that if $x \in R$ is as in the Superficial Element Lemma, then $e(I/xR) = e(I)$. We also noted that if $I \subseteq R$ is an $\mathfrak{m}$-primary ideal and $J := (0 : I^\infty)$, then $e(I) = e((I + J)/J)$, for local ring of positive dimension, and that the image if I in R/J has positive grade. This then led to the following theorem:

Theorem. let R be a local ring and $I \subseteq R$ an ideal generated by a system of parameters. Then:

(i) $e(I) \leq \lambda(R/I)$

(ii) $e(I) = \lambda(R/I)$ if and only if R is Cohen-Macaulay.

The proof of (i) and the 'if' direction followed in a straightforward way by using induction, the superficial element lemma, and the use of $J = (0 : I^\infty)$. The proof of the only if direction of (ii) given in class was incomplete. Here is a complete proof. Suppose $e(I) = \lambda(R/I)$, with J as before. Then,

$$e(I) = \lambda(R/I) \geq \lambda(R/(I + J)) \geq e((I + J)/J) = e(I),$$

thus in R/J, $e((I+J)/J) = \lambda(R/(I+J)$. Now assume that R/J is Cohen-Macaulay. The displayed expression above also shows that $\lambda(R/I) = \lambda(R/(I + J))$, from which it follows that $\lambda((I + J)/J) = \lambda(J/(I \cap J)) = 0$. Thus, $J = I \cap J$. Since the generators of I form a regular sequence modulo J (as R/J is Cohen-Macaulay), we have $I \cap J = IJ$, and thus $J = IJ$. By Nakayama's lemma, $J = 0$ and thus, R is Cohen-Macaulay. We now prove the required implication by induction on d. Suppose $d = 1$. By the preceding statements we can reduce to the case that $J = 0$, in which case R is already Cohen-Macaulay. Now suppose $d > 1$. Again, it suffices to assume $J = 0$, so $\mathrm{grade}(I) > 0$. Then there exists a non-zerodivisor $x \in I$ such that $I = \langle x, x_2, \ldots, x_d \rangle$ and $e(I') = \lambda(R'/I')$, where R' denotes modulo $\langle x \rangle$. By induction R' is Cohen-Macaulay, therefore R is Cohen-Macaulay.

Wednesday, November 2. We stated and proved the following theorem, modulo a result concerning *superficial elements*.

Theorem. Let $(R, \mathfrak{m})$ be a d-dimensional local ring and $I \subseteq R$ an ideal. Then there exits positive integers $e_0, \ldots, e_d$ such that $e_0 > 0$ and

$$\lambda(R/I^{n+1}) = e_0\binom{n+d}{d} + e_{d-1}\binom{n+d-1}{d-1} + \cdots + e_d,$$

for $n >> 0$.

The main idea of the proof was to first reduce to the case that $\mathrm{grade}(I) > 0$ and then proceed by induction using a superficial element to reduce to the $d-1$ case by noting that $\lambda(R/\langle I^{n+1}, x\rangle) = \lambda(R/I^{n+1}) - \lambda(R/I^n)$, for $n >> 0$. We noted that the theorem shows that for $n >> 0$, $\lambda(R/I^{n+1})$ is governed by a polynomial of the form $P_I(n) = \frac{e_0}{d!}n +$ lower order terms. $P_I(n)$ is the *Hilbert-Samuel* polynomial of I and $e(I) := e_0$ is the *multiplicity* of I. It followed that $e(I) = \lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(R/I^{n+1})$.

We ended class by partially proving the following:

Superficial Element Lemma. Let $(R, \mathfrak{m})$ be a local ring with infinite residue field and $I \subseteq R$ an $\mathfrak{m}$-primary ideal having positive grade. Then there exists $x \in I$ such that:

(i) x is a non-zerodivisor

(ii) x is a minimal generator of I

(iii) $\lambda(R/\langle I^{n+1}, x\rangle) = \lambda(R/I^{n+1}) - \lambda(R/I^n)$, for $n >> 0$.

Monday, October 31. We finished the proof of the Auslander-Buchsbaum formula, namely the case where $\text{depth}(R) > 0$ and $\text{depth}(M) = 0$. The key point was that by letting K be the kernel of the map in a minimal presentation of M, $\text{pd}(K) = \text{pd}(M) - 1$ and $\text{depth}(K) = 1 = \text{depth}(M) + 1$, so the remaining case reduces to the second case presented in the previous lecture. We immediately recorded two corollaries of the Auslander-Buchsbaum formula:

Corollaries. 1. Let $(R, \mathfrak{m})$ be a regular local ring and $I \subseteq R$, an ideal. Then R/I is Cohen-Macaulay if and only if $\text{pd}(R/I) = \text{height}(I)$.

2. Let R be a Noetherian ring and $I \subseteq R$ an ideal.
   (a) $\text{grade}(I) \leq \text{pd}(R/I)$.
   (b) Suppose $\text{grade}(I) = \text{pd}(R/I)$, so that I is a *perfect ideal*. Then $\text{grade}(P) = \text{grade}(I)$, for all $P \in \text{Ass}(R/I)$. Thus, *perfect ideal are grade unmixed.*

We ended class by proving the:

Hilbert-Burch Theorem. Let R be a Noetherian ring and I a grade two ideal. Then I is a perfect ideal if and only if $I = I_n(A)$, for A an $(n+1) \times n$ matrix over R. In particular, if R is a regular local ring and $I \subseteq R$ is a height two ideal, then R/I is Cohen-Macaulay if and only if $I = I_n(A)$, for some $(n+1) \times n$ matrix A with entries in R.

For the idea of the proof: If $I = I_n(A)$, then the Buchbaum-Eisenbud exactness theorem shows that

$$(**) \qquad 0 \to R^n \xrightarrow{A} R^{n+1} \xrightarrow{\phi} R \to R/I \to 0$$

is a free resolution of R/I, where ϕ takes the standard basis elements of R^{n+1} to the (appropriately) signed minors of A. For the converse, if $I = \langle x_1, \ldots, x_{n+1}\rangle$, and (**) is a resolution of R/I, where now ϕ takes the ith standard basis element of R^{n+1} to x_i, one argues that $x_i\delta_j = \pm x_i\delta_j$, for all $i \neq j$, with δ_j the minor of A obtained by deleting the jth row of A. If we let $y = t_1\delta_1 + \cdots + t_{n+1}\delta_{n+1}$ be a non-zerodivisor in $I_n(A)$, then there exists $t \in R$ such that $yx_i = t\delta_i$ for all i. Thus, $yI = tI_n(A)$, so in particular, $tI_n(x) \subseteq \langle y\rangle$. Since $I_n(A)$ has grade at least two (by the Buchsbaum-Eisenbud theorem), $t = r_0y$, for some $r_0 \in R$. Thus, $I = r_0I_n(A)$. Since I has grade two, r_0 must be a unit, so $I = I_n(A)$.

Friday, October 18. We began class by showing that if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring and $I \subseteq R$ is an ideal, then $\dim(R) = \dim(R/I) + \text{height}(I)$. We then used the ideas in the proof of this result to show that a Noetherian Cohen-Macaulay ring is universally catenary. This was followed by a brief homological aside, where we defined what it means for a finitely generated module over a local ring to have finite projective dimension ($\text{pd}(M) < \infty$) and noted (but did not prove) that over such a ring $pd(M) < \infty$ if and only if M admits a *minimal free resolution* of length r. We then sketched a proof using the Tor functor showing that every module over a regular local ring has finite projective dimension, using the fact (essentially established previously) that the Koszul on a minimal generating set for the maximal ideal gives a free resolution of the residue field. The purpose of this discussion is be able to show in the next lecture that if R is a regular local ring and $I \subseteq R$ is an ideal, then the local ring R/I is Cohen-Macaulay if and only if $\text{pd}(R/I) = \text{height}(I)$.

We ended class by stating and partially proving the fundamental:

Auslander-Buchsbaum Formula. Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module with finite projective dimension. Then:

$$\text{depth}(R) = \text{depth}(M) + \text{pd}(M).$$

We provided a proof of the cases when $\text{depth}(R) = 0$ and both R and M have positive depth. The proof of the theorem will be completed in the next lecture.

Wednesday, October 26. We continued our discussion of Cohen-Macaulay rings by proving the following results:

Proposition. Let $(R, \mathfrak{m})$ be a local ring of dimension d. The following are equivalent:

(i) R is Cohen-Macaulay.

(ii) Some system of parameters forms a regular sequence.
(iii) Every system of parameters forms a regular sequence.

This proposition had the the following consequences, presented as corollaries:

(i) A regular local ring is Cohen-Macaulay.
(ii) If $(R, \mathfrak{m})$ is Cohen-Macaulay, $I := \langle x_1, \ldots, x_t \rangle$, with height$(I) = t$, then $x_1, \ldots, x_t$ form a regular sequence.
(iii) For R and I as in (ii), R/I is Cohen-Macaulay.
(iv) If R is Noetherian and Cohen-Macaulay, then the polynomial ring $R[x]$ is Cohen-Macaulay.

We were then able to prove the:

Unmixedness Theorem for Cohen-Macaulay rings. Let R be a Noetherian ring. The following are equivalent.

(i) R is Cohen-Macaulay.
(ii) height$(P) = t$ whenever $P \in \text{Ass}(R/I)$ and I is a height t ideal generated by t elements.

The substantial part of the proof is (i) implies (ii), a sketch of which follows. Maintaining the notation from the statement of the theorem we can assume R is Cohen-Macaulay, local at P, and show R has dimension t. By (ii) above I is generated by a regular sequence. By the corollary presented at the end of the previous lecture, $\dim(R/P) = \dim(R) - t$. But R is local, so $0 = \dim(R) - t$, which gives what we want.

We ended class by defining the concepts of *catenary* and *universally catenary* for Noetherian rings.

Monday, October 24. We began our discussion of Cohen-Macaulay rings with the following theorem.

Theorem. Let R be a Noetherian ring. The following are equivalent:

(i) grade$(\mathfrak{m})$ = height$(\mathfrak{m})$, for all maximal ideals $\mathfrak{m} \subseteq R$.
(ii) grade(P) = height(P), for all primes ideals $P \subseteq R$.
(iii) grade(I) = height(I), for all ideals $I \subseteq R$.

R is said to be *Cohen-Macaulay* if any one, and hence all, of the conditions above hold. The proof of the theorem required showing that (i) implies (iii), and this was done by assuming there was an ideal I maximal with respect to the property of having grade$(I) <$ height(I) - and constructing a larger ideal whose grade is less than its height. This was possible, since the hypothesis (i) enables one to extend the maximal regular sequence in I to a longer regular sequence, so that any prime associated to the new sequences fails to have its grade equal to its height. A key point, used in this proof and a subsequent proof, was the following:

Lemma. Let $x \in R$ be a non-zerodivisor and $P \in \text{Ass}(R)$. If $Q \subseteq R$ is a prime ideal minimal over $\langle P, x \rangle$, then $Q \in \text{Ass}(R/xR)$.

As a corollary to the theorem we showed that the following are equivalent: (a) R is Cohen-Macaulay; (b) $R_{\mathfrak{m}}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m} \subseteq R$; (c) R_P is Cohen-Macaulay for all prime ideals $P \subseteq R$.

We then used the lemma above to prove:

Proposition. Let $(R, \mathfrak{m})$ be a local ring. Then depth$(R) \leq \dim(R/P)$, for all $P \in \text{Ass}(R)$.

The proposition then lead to the following important consequence, which is part of the unmixedness property for Cohen-Macaulay rings:

Corollary. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim(R) = d$ and $\underline{x} = x_1, \ldots, x_t$ a regular sequence. Then $\dim(R/P) = d - t$, for all $P \in \text{Ass}(R/\langle \underline{x} \rangle)$.

Friday, October 21. Today's lecture was devoted to two applications of the Buchsbaum-Eisenbud theorem. Given a sequence of elements $\underline{x} = x_1, \ldots, x_n$ in the Noetherian ring R, we defined $\mathcal{K}(\underline{x})$, the Koszul complex on $\underline{x}$, to be the complex of finitely generated free R-modules

$$\mathcal{K}(\underline{x}): \quad 0 \longrightarrow K_n \xrightarrow{\phi_n} K_{n-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{\phi_1} K_0,$$

where, for each $0 \leq r \leq n$, $K_r := R^{\binom{n}{r}}$, with standard basis labelled $e_{i_1} \wedge \cdots \wedge e_{i_r}$, with $1 \leq i_1 < \cdots i_r \leq n$, for $1 \leq r \leq n$ and $K_0 = R$, with basis $e_0 = 1$, with

$$\phi_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \Sigma_{j=1}^r (-1)^{1+j} x_j e_{i_1} \wedge \overset{\hat{j}}{\cdots} \wedge e_{i_r}.$$

We first showed explicitly, that when $\underline{x}$ is a regular sequence and $n = 3$, $\mathcal{K}(\underline{x})$ satisfies the conditions required in the Buchsbaum-Eisenbud theorem insuring that $\mathcal{K}(\underline{x})$ is exact. We then stated and sketched a proof of the following **fundamental** proposition.

Proposition. Let R be a Noetherian ring, and $\underline{x} = x_1, \ldots, x_n$ a regular sequence. Then $\mathcal{K}(\underline{x})$ is exact.

Before proving the proposition, we noted the general fact that for any sequence $\underline{x}$, $x_i \cdot \mathrm{H}_j(\mathcal{K}(\underline{x})) = 0$, for all i and j, where $\mathrm{H}_j(\mathcal{K}(\underline{x}))$ denotes the jth homology module in $\mathcal{K}(\underline{x})$. We left the proof of this fact as an exercise. For the proof of the proposition, we first noted that if we localize $\mathcal{K}(\underline{x})$ at the set S of non-zerodivisors, the fact just stated shows that $(\mathcal{K}(\underline{x}))_S$ is exact (in fact, split exact), so that the proposition from the previous lecture shows that $\mathcal{K}(\underline{x})$ satisfies the rank condition and $\mathrm{grade}(I(\phi_i)) > 0$, for $1 \leq i \leq n$. Similarly, if P is a prime ideal satisfying $\mathrm{depth}(R_P) \leq n-1$, then $(\mathcal{K}(\underline{x}))_P$ is split exact. Thus, for each $1 \leq i \leq n$, the image of $(\phi_i)_P$ is a free summand of K_{i-1} and hence $I(\phi_i \not\subseteq P$. This implies that each $I(\phi_i)$ has grade at least n, so that the Buchsbaum-Eisenbud theorem shows $\mathcal{K}(\underline{x})$ is exact. (Note: In fact, each $I(\phi_i) = I^r$, for $r = \mathrm{rank}(\phi_i)$, and thus each $I(\phi_i)$ has grade equal to n.)

We ended class by using the Buchbaum-Eisenbud theorem to show that if S is a UFD, $A = (X_{ij})$ is the generic $(n+1) \times n$ matrix over S, and $\delta_1, \ldots, \delta_{n+1}$ are the signed $n \times n$ minors of A, then

$$0 \longrightarrow R^n \xrightarrow{A} R^{n+1} \xrightarrow{\delta_1, \cdots, \delta_{n+1}} R$$

is an exact sequence. In particular, if $I := \langle \delta_1, \ldots, \delta_{n+1} \rangle$, then

$$0 \longrightarrow R^n \xrightarrow{A} R^{n+1} \xrightarrow{\delta_1, \cdots, \delta_{n+1}} R \to R/I \to 0$$

is a free resolution of R/I.

Wednesday, October 19. We continued with our discussion of the Buchsbaum-Eisenbud Exactness theorem, culminating in the proof of the theorem. As preliminaries, we first stated the following lemma, whose proof we did not give in class.

Lemma. Let (R, P) be a local ring and $\phi : R^n \to R^m$ an R-module homomorphism of free R-modules. Suppose $r \geq 1$, $v_1, \ldots, v_r \in \mathrm{im}(\phi)$, $V := \langle v_1, \ldots, v_r \rangle$ and A is the $m \times r$ matrix whose columns are $v_1, \ldots, v_r$. The following are equivalent:

(i) $v_1, \ldots, v_r$ extend to a basis for R^m.
(ii) V is a free summand of R^m.
(iii) There exists an $r \times r$ minor of A not in P.

Moreover, if the statements above hold and ϕ has rank r, then $\mathrm{im}(\phi)$ is a free rank r summand of R^m.

Proof. That (i) implies (ii) is clear. Suppose (ii) holds. We can expand A to an invertible $m \times m$ matrix B whose first r columns are $v_1, \ldots, v_r$. Then $|B|$ is not in P. Since $|B| \in I_r(\phi)$, we have $I_r(\phi) \not\subseteq P$, which is what we want. Suppose (iii) holds. Suppose some $r \times r$ minor δ of A is not in P. Then, letting ' denote modulo P, we have that $v_1', \ldots, v"r'$ can be extended to a basis $v_1', \ldots, v_r', w_{r+1}', \ldots w_m'$ for the vector space $(R/P)^m$. It follows that $R^m = \langle v_1, \ldots, w_m \rangle$ (by Nakayama's lemma). Writing the standard basis for R^m in terms of these column vectors shows that there exists an $m \times m$ matrix Q such that DQ the identity matrix, where D is the matrix with columns $v_1, \ldots, w_m$. It follows that D is an invertible matrix, so that $v_1, \ldots, w_m$ is a basis for R^m, which gives (i). Finally, if the conditions hold, and ϕ has rank r, then for any column C of ϕ, and any $r \times r$ minor δ of A, the preliminary fact from the previous lecture shows that $\delta \cdot C$ is in V. Since some δ is a unit, it follows that C is contained in V and thus $\mathrm{im}(\phi) = V$ is a free rank r summand of R^m.

The statement of the lemma above was followed by a proof of the next proposition:

Proposition. Let R be a Noetherian ring. Given the complex

$$\mathcal{F}: \qquad 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} F_{n-1} \to \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

of finitely generated free R-modules, and $S \subseteq R$ the set of non-zerodivisors. Then $\mathcal{F}_S$ is exact if and only the rank condition holds for $\mathcal{F}$ and $I(\phi) \cap S = \emptyset$, for $1 \leq i \leq n$.

The proof of the proposition was by induction on n, with McCoy's theorem handling the case $n = 1$. The key points were using the lemma above together with the observation that when $I(\phi)$ contains a non-zerodivisor, ϕ and ϕ_S have the same rank, for any multiplicatively closed set.

We finished class with the proof of the full version of the Buchasbaum-Eisenbud theorem. Again, this proof was by induction on n, with McCoy's theorem as the base case, together with the previous proposition. Another key point in both directions of the proof was that certian complexes of length $n-1$ obtained from $\mathcal{F}$ by modding out a non-zerodivisor were exact.

Monday, October 17. We continued our discussion that will lead to a proof the Buchsbaum-Eisenbud exactness theorem. This started with the presentation of several preliminary facts about matrices and determinants over a commutative ring. We did not prove these facts. Aside from the definitions of the classical adjoint $\tilde{A}$ of an $n \times n$ matrix and the fact that $\tilde{A} \cdot A = |A| \cdot I_n = A \cdot \tilde{A}$, the most important fact we cited was the following. Suppose A is an $m \times n$ matrix over R of rank r and A_0 is an $m \times r$ submatrix of A. Then for any $r \times r$ minor δ of A_0, and any column C of A, we have $\delta \cdot C = \delta_1 \cdot C_1 + \cdots + \delta_r \cdot C_r$, where $C_1, \ldots, C_r$ are the columns of A_0 and δ_i is the $r \times r$ minor obtained from A_0 by first replacing the ith column of A by C, and then deleting the same rows from this matrix that were deleted from A_0 to obtain δ.

The preliminaries were followed by the following: Let A be an $n \times n$ matrix with entries in R. Then the map $R^n \xrightarrow{A} R^n$ is injective if and only if $|A|$ is a non-zerodivisor. This is a special case of the following version of McCoy's theorem, which serves as the base case for an induction proof of the Buchsbaum-Eisenbud Theorem:

McCoy's Theorem for Noetherian rings. Let R be a Noetherian ring and $0 \to R^n \xrightarrow{A} R^m$ a complex of length one, given by the $m \times n$ matrix A. Then the complex is exact if and only if $n \leq m$ and $I_n(A)$ has grade at least one.

The proof of this theorem was by induction on n, using the observation that if the entries of A belong to $P \in \text{Ass}(R)$ and $P = (0 : c)$, then $A \cdot v = \vec{0}$, where $\vec{0} \neq v = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix}$.

Friday, October 14. We began class with the question that encapsulates the title of a celebrated paper of Buchsbaum-Eisenbud. Let R be a Noetherian ring. Given the complex

$$(**) \qquad 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} F_{n-1} \to \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

of finitely generated free R-modules, what makes (**) exact? The view expressed in this lecture is that an answer can be inferred by considering some special cases. So, we worked through the following cases:

1. When R is a field, we noted that the *Rank plus Nullity theorem* yields (**) is exact if and only if $\text{rank}(F_i) = \text{rank}(\phi_{i+1}) + \text{rank}(\phi_{i+1})$ for $0 \leq i \leq n-1$ and $\text{rank}(F_n) = \text{rank}(\phi_n)$. In this case we say that (**) satisfies the *rank condition*. Here $\text{rank}(F_i)$ denotes the dimension of F_i and $\text{rank}(\phi_i)$ equals the rank of any matrix representing ϕ_i. In fact, we will assume that the ϕ_i are just matrices. We then noted that in the general case, $\text{rank}(F_i)$ will denote the number of elements in a basis for F_i and $\text{rank}(\phi_i)$ denotes the largest positive integer t such that ϕ_i has a $t \times t$ submatrix with non-zero determinant.

2. We noted that if R is an integral domain and (**) is exact, then the rank condition from 1 holds for (**), since (**) remains exact upon passing to the quotient field of R and neither then ranks of the free modules or nor the ranks of the maps in the complex change upon passing to the quotient field of R.

3. When $n = 1$, exactness of $(**)$ is essentially McCoy's Theorem, which states: *Let A be an $m \times n$ matrix over the commutative ring S. Then the homogeneous system of m equations in n unknowns with coefficient matrix A has a non-trivial solution if and only if there exists $0 \neq c \in S$ such that $c \cdot \delta = 0$, for all $n \times n$ minors of A.* Equivalently, (**) is exact if and only if there does not exist $0 \neq c \in R$ such that $c \cdot I_n(\phi_1) = 0$,

where $F_1 = R^n$ and $I_n(\phi_1)$ denotes the ideal of R generated by the $n \times n$ minors of ϕ_1. Since R is Noetherian, this is equivalent to the statement: When $n = 1$, (**) is exact if and only if $\text{grade}(I_n(\phi_1)) > 0$.

4. When R is an integral domain, $n = 2$ and (**) is exact, we showed that the rank condition stated in 1 holds and that $I_n(\phi_2)$ has grade at least two, where $n = \text{rank}(F_2)$. We also noted that $I_r(\phi_1)$ has grade at least one, where r is the rank of ϕ_1, since R is an integral domain.

5. When R is an integral domain, $n = 2$, we showed that if the rank condition holds and $I_n(\phi_2)$ has grade at least two (where $n = \text{rank}(F_2)$), then (**) is exact.

This then lead to a statement of the general theorem. In the statement of the theorem, we write $I(\phi_i)$ for the ideal of $t \times t$ minors of ϕ_i, where t is the rank of ϕ_i.

Buchsbaum-Eisenbud Exactness Theorem. Let R be a Noetherian ring and

$$(**) \qquad 0 \longrightarrow F_n \xrightarrow{\phi_n} F_n \xrightarrow{\phi_{n-1}} F_{n-1} \to \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

a complex of finitely generated free R-modules. Then (**) is exact if and only the following conditions hold:

(i) $\text{rank}(F_i) = \text{rank}(\phi_i) + \text{rank}(\phi_{i+1})$ for $0 \leq i \leq n-1$ and $\text{rank}(F_n) = \text{rank}(\phi_n)$.
(ii) Either $I(\phi_i) = R$ or $\text{grade}(I(\phi_i)) \geq i$, for all $1 \leq i \leq n$.

We ended class with the observation that our approach to the theorem is via induction with McCoy's theorem serving as the base case - together with the comment that the proof cannot be simplified by assuming R is an integral domain, since the rings encountered in an inductive step may not be integral domains, even if R is an integral domain to begin with.

Wednesday, October 12. For an ideal I contained in the Noetherian ring R, we defined what it means for the sequence $\underline{x} = x_1, \ldots, x_t \in I$ to be a *maximal regular sequence from* I, namely $\underline{x}$ is a regular sequence and $I \subseteq P$ for some $P \in \text{Ass}(R/\langle \underline{x} \rangle)$, or equivalently, there does not exist $x_{t+1} \in I$ such that $\underline{x}, x_{t+1}$ is a regular sequence. We then proved two standard facts: (i) If $\underline{x} \subseteq I$ is a regular sequence, then $\text{height}(I) \geq t$ and (ii) if $\underline{x} \subseteq R$ is a regular sequence contained in the Jacobson radical of R, then any permutation of the elements $x_1, \ldots, x_t$ remains a regular sequence. This latter fact was a needed in the proof of the important:

Theorem. Let R be a Noetherian ring and $I \subseteq R$ an ideal. Then all maximal regular sequences from R have the same length.

For the proof of the theorem, we noted that it is enough to prove the following statement: Suppose $\underline{x}$ and $\underline{y} = y_1, \ldots, y_s$ are maximal regular sequences from I with $t \leq s$. Then, for $P \in \text{Ass}(R/\langle \underline{x} \rangle)$ with $I \subseteq P$, $P \in \text{Ass}(R/\langle y_1, \ldots, y_t \rangle)$. We proved this statement by induction on t, after localizing at P. Upon doing so, all of the regular sequences can be permuted. For $t > 1$, one then finds $z \in I$ such that $x_1, \ldots, x_{t-1}, z$ and $y_1, \ldots, y_{t-1}, z$ are regular, and thus, $z, x_1, \ldots, x_{t-1}$ and $z, y_1, \ldots, y_{t-1}$ form regular sequences. Induction applied to $R/\langle z \rangle$ gives $P \in \text{Ass}(R/\langle z, y_1, \ldots, y_{t-1} \rangle)$ which ultimately gives $P \in \text{Ass}(R/\langle y_1, \ldots, y_t \rangle)$.

The theorem above enables one to define the *grade of* I to be the length of any maximal regular sequence from I, denoted $\text{grade}(I)$. We then noted that if $(R, \mathfrak{m})$ is a local ring, $\text{grade}(\mathfrak{m})$ is called the *depth of* R. We also noted that one can extend the notions of grade and depth to modules: For an ideal $I \subseteq R$ and a finitely generated R-module M, one defines, in a similar fashion, $\text{grade}(I, M)$, *grade of* I *on* M to be the length of any maximal regular sequence on M from I and the *depth of* M to be $\text{grade}(\mathfrak{m}, M)$, when $(R, \mathfrak{m})$ is local.

Friday October 7. We began class with a preliminary discussion of the Frobenious homomorphism $F : R \to R$, where R is a Noetherian, commutative ring containing a field of characteristic $p > 0$, given by $F(r) = r^p$. Its eth iterate $F^e : R \to R$, given by $F^e(r) = r^{p^e}$, for all $e \geq 1$ makes R into an R-module ${}^eR$. We then sketched a proof of one direction of the celebrated theorem of Kunze, which states that a local ring R containing a field of characteristic p is a regular local ring if and only if F is flat - namely showing that R regular implies F is flat. We then used this to prove a proposition showing that for R regular as before, $\text{Ass}(R/I) = \text{Ass} R/I^{[q]}$, for all ideals $I \subseteq R$ and $q = p^e$. We were then able to give a a proof of the following theorem, due in Hochster and Huneke in positive characteristic and Ein-Lazarsfled-Smith, for rings essentially of finite type over $\mathbb{C}$:

Theorem. Let R be a regular local ring containing a field of characteristic $p > 0$. Suppose $P \subseteq R$ is a prime ideal with height$(P) = c$. Then $P^{(cn)} \subseteq P^n$, for all $n \geq 1$. In particular, if $d = \dim(R)$, then $P^{((d-1)n)} \subseteq P^n$, for all $n \geq 1$ and all prime ideals P.

A crucial, and interesting, part of the proof required showing that $P^{(cq)} \subseteq P^{[q]}$. This can be done by localizing at primes in the set $\mathrm{Ass}(R/P^{[q]})$, which by the proposition alluded to above, is just $\{P\}$. But then $P^{(cq)}$becomes P^{cq}. After localizing at P, P is generated by c elements $x_1, \ldots, x_c$, so that P^{cn} is generated by the monomials of degree cq in these elements, and thus each such monomial contains at least a qth power of some x_i, showing that $P^{cn} \subseteq P^{[q]}$.

Wednesday, October 5. We began by recalling that for any prime P in a Noetherian ring R, there exists an ideal $J \subseteq R$ such that $P^{(n)} = (P^n : J^\infty)$ for all $n \geq 1$. Thus, the question of when the P-symbolic topology is equivalent to the P-adic topology is answered by the following theorem of P. Schenzel:

Theorem. Let R be a Noetherian rings and $I, J \subseteq R$ ideals. Then the topologies defined by $\{(I^n : J^\infty)\}_n$ and $\{I^n\}_n$ are equivalent if and only if for all $Q \in V(I) \cap A^*(I)$, $I\widehat{R_Q} + z$ is not $Q\widehat{R_Q}$-primary for all $Q \in V(J) \cap A^*(I)$ and $z \in \mathrm{Ass}(\widehat{R_Q})$, where $A^*(I) = \bigcup_{n \geq 1} \mathrm{Ass}(R/I^n)$.

Using the fact that the topologies defined by $\{(I^n : J^\infty)\}_n$ and $\{I^n\}_n$ are equivalent if and only if for all $Q \in V(I) \cap A^*(I)$, the topologies defined by $\{(I^n : J^\infty)_Q\}_n$ and $\{I^n_Q\}_n$ are equivalent for all $Q \in V(J) \cap A^*(A)$, the theorem followed immediately from the following three facts, presented as a sequence of propositions:

(i) The topologies defined by filtrations $\{(I^n : J^\infty)\}_n$ and $\{I^n\}_n$ are equivalent if and only if for all $Q \in V(I) \cap A^*(I)$, the topologies defined by $\{(I^n : Q^\infty)_Q\}_n$ and $\{I^n_Q\}_n$ are equivalent.

(ii) For all $Q \in V(J) \cap A^*(A)$, the topologies defined by $\{(I^n : Q^\infty)_Q\}_n$ and $\{I^n_Q\}_n$ are equivalent if and only if for all $Q \in V(J) \cap A^*(A)$, the topology defined by $\{(I^n : Q^\infty)_Q\}_n$ is finer than the topology defined by $\{Q^n_Q\}_n$.

(iii) For all $Q \in V(J) \cap A^*(A)$, the topology defined by $\{(I^n : Q^\infty)_Q\}_n$ is finer than the topology defined by $\{Q^n_Q\}_n$ if and only if for all $Q \in V(J) \cap A^*(A)$, $I\widehat{R_Q} + z$ is not $Q\widehat{R_Q}$-primary.

The proofs of (i) and (ii) followed from the hypotheses and and fairly straightforward arguments with primary decomposition. The proof of (iii), the most significant of the items above, followed from Chevalley's theorem (applied in the rings $\widehat{R_Q}$) and the consequence of the Artin-Rees theorem which states for an ideal $C \subseteq R$, there exists $k \geq 1$ such that $(x : C^n) \subseteq (0 : x) + C^{n-k}$, for $n \geq k$.

Monday, October 3. We began class by stating the following very interesting theorem of C. Huneke: *Let R be a three-dimensional regular local ring and $P \subseteq R$ a height two prime ideal. If $P^{(n)} = P^n$ for some $n \geq 2$, then P is generated by a regular sequence.* This forms a strong converse to the theorem presented in the previous lecture (for the given set of prime ideals). We noted that the proof of this theorem and a few others depend upon multiplicity theory. For example, Nagata has shown that if (R, M) is a regular local ring, and $P \subseteq R$ is a prime ideal, then $P^{(n)} \subseteq M^n$, for all n. This is a purely algebraic version of a key component of the Zariski-Nagata theorem, and this result also uses multiplicity theory, thus suggesting a connection between multiplicities and symbolic powers. We then discussed what it means for two ideal topologies to be equivalent. This was followed by noting that if P is a prime ideal, there there exists an ideal J such that $P^{(n)} = (P^n : J^\infty)$ for all n, where for any two ideals $I, J \subseteq R$, $(I : J^\infty) = (I : J^s)$, where $(I : J^s)$ denotes the stable value of the ascending chain $(I : J) \subseteq (I : J^2) \subseteq \cdots$. We then noted that our next goal was to determine when the I-adic topology is equivalent to the $\{(I^n : J^\infty)\}$ topology.

We ended class by stating and proving the following theorem of Chevalley:

Theorem. Let (R, M) be a complete local ring and $\{I_n\}$ a collection of ideals satisfying: (i) $I_{n+1} \subseteq I_n$ for all n and (ii) $\bigcap_{n \geq 1} I_n = 0$. Then for each $n \geq 1$, there exists $s(n)$ such that $I_{s(n)} \subseteq M^n$.

Friday, September 30. We spent the class giving (almost all of) the details showing that if P is a prime ideal generated by a regular sequence, then $P^{(n)} = P^n$ for all $n \geq 1$. This is equivalent to showing that $\mathrm{Ass}(R/P^n) = P$ for all n. For this we noted it was enough to prove that if I is an ideal in a Noetherian ring R generated by a regular sequence, then I^n/I^{n+1} is a free R/I-module for all $n \geq 1$. This latter fact in turn followed from the following proposition:

Proposition. If $\underline{x} = x_1, \dots, x_t$ is a regular sequence and $F(Y_1, \dots, Y_t)$ is a homogeneous polynomial such that $F(\underline{x}) = 0$, then $F(Y_1, \dots, Y_t)$ has coefficients in $I := \langle \underline{x} \rangle$.

The proof of the proposition was by induction on t and $n := \deg(F)$, done in tandem with the following statement: If $z \in R$ satisfies $(I : z) = I$, then $(I^n : z) = I^n$, for all $n \geq 1$.

Wednesday, September 28. We began class by finishing the second part of the last lemma from the previous lecture in the following form: If $A \subseteq B$ are finitely generated modules over the Noetherian ring R and $\underline{x} \subseteq R$ is a sequence of elements forming a regular sequence on B/A, then $\underline{x}B \cap A = \underline{x}A$. We then proved the standard result: If $N \subseteq M$ are finitely generated modules over the Noetherian ring R, then $N = M$ if and only if $N_P = M_P$, for all $P \in \text{Ass}(M/N)$.

With the preliminaries out of the way, we then stated and proved the Eisenbud-Hochster *Nullstellensatz with nilpotents* theorem as given in their original paper:

Theorem. Let R be a Noetherian ring, $P \subseteq R$ a prime ideal, $n \geq 1$ and A a finitely generated R-module. Let $\mathcal{C}$ denote the maximal ideals $M \subseteq R$ containing P such that $(R/P)_M$ is a regular local ring. Assume:

(i) $P = \bigcap_{M \in \mathcal{C}} M$.
(ii) $\text{Ass}(A) = \{P\}$.
(iii) P^n annihilates A.

Then $0 = \bigcap_{M \in \mathcal{C}} M^n A$.

In the last lecture we noted that this theorem applied in the case $A = R/P^{(n)}$ gives the Zariski-Nagata theorem when R is a polynomial ring in finitely many variables over an algebraically closed field characteristic zero.

Here is a sketch of the proof of the theorem above: One sets $A_i := P^i A$, for $0 \leq i \leq n$ and considers the filtration $A = A_0 \supseteq A_2 \supseteq \cdots \supseteq A_n = (0)$. The quotients A_i/A_{i+1} are free over R/P upon localizing at P and hence are free over $(R/P)_f$ upon inverting f, for some $f \in R \backslash P$. One notes that we may then reduce to the case that each A_i/A_{i+1} is free over R/P. The hypothesis on P shows $0 = \bigcap_{M \in \mathcal{C}} M(A_i/A_{i+1})$ and one then notes it suffices to prove $M^n A \cap A_i \subseteq MA_i$, for all $M \in \mathcal{C}$. From here, one may localize at M as well. Then $M = P + \underline{x}$, where $\underline{x}$ is regular on A_j/A_{j+1}, for all j. The lemmas from the last lecture give that $\underline{x}$ is regular on A/A_i and thus $\underline{x}A \cap A_i = \underline{x}A_i$. From $M = P + \underline{x}$, once deduces $M^n A \cap A_i \subseteq MA_i$ since $M^n \subseteq P^n + \underline{x}$ and $P^n \cdot A = 0$.

Monday, September 26. We continued with the following setting: $R = k[x_1, \dots, x_d]$, where k is an algebraically closed field of characteristic zero and $P \subseteq R$ is a prime ideal. We let $X = V(P)$ be the irreducible variety in k^d defined by P and $P^{\langle n \rangle}$, as before, denote the elements $f \in R$ such that $D_r(f)(\underline{\alpha}) = 0$, for all $\underline{\alpha} \in X$ and all mixed partial differential operators D_r of order $0 \leq r < n$, equivalently, $P^{\langle n \rangle}$ consists of all $f \in R$ with $D_r(f) \in P$, for $0 \leq r < n$. We also use $M_{\underline{\alpha}}$ to denote the maximal ideal $\langle x - \alpha_1, \dots, x_d - \alpha_d \rangle$, equivalently, $M_{\underline{\alpha}}$ consists of those $f \in R$ such that $f(\underline{\alpha}) = 0$.

We then proved the following:

Proposition. Let R and $P \subseteq R$ be as in our general setting. Then, for $n \geq 1$,

$$P^{\langle n \rangle} = \bigcap_{\underline{\alpha} \in X} M_{\underline{\alpha}}^n = \bigcap \{M^n \mid M \text{ is a maximal ideal containing } P\}.$$

We then recorded the statements of the Eisenbud-Hochster *Nullstellensatz with nilpotents* theorem:

Theorem. Let R be a Noetherain ring and $P \subseteq R$ be a prime ideal such that $P = \bigcap_{M \in \mathcal{C}} M$, where $\mathcal{C}$ is a set of maximal ideals containing P such that $(R/P)_M$ is a regular local ring. Then $\bigcap_{M \in \mathcal{C}} M^n \subseteq P^{(n)}$, for all $n \geq 1$.

We then noted that the Zariski-Nagata theorem follows immediately from this:

Proof of Zariski-Nagata. Let R and P be as in our general set up. From the lecture of September 12, we have that for all $P \subseteq R$, $P = \bigcap_{M \in \mathcal{C}} M$, where $\mathcal{C}$ is the set of maximal ideals containing P such that

$(R/P)_M$ is a regular local ring for all $M \in \mathcal{C}$ so that the Eisenbud-Hochster theorem applies to all $P \subseteq R$. Now using the proposition from the end of the previous lecture, and the proposition above, we have,

$$P^{(n)} \subseteq P^{\langle n \rangle} = \bigcap \{M^n \mid M \text{ is a maximal ideal containing } P \} \subseteq \bigcap_{M \in \mathcal{C}} M \subseteq P^{(n)}$$

and thus $P^{(n)} = P^{\langle n \rangle}$, which gives the result.

We finished class with the following Lemmas:

Lemma. Let R be a Noetherian ring and A a finite R-module. Suppose $P \subseteq R$ is a prime ideal such that A_P is a free R_P-module. Then there exists $f \in R \backslash P$ such that A_f is a free R_f-module.

Lemma. Let R be a Noetherian ring and M a finite R-module. Let $M_r \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$ be a filtration of submodules and $\underline{x} = x_1, \ldots, x_t$ be a regular sequence on each M_{i+1}/M_i. Then $\underline{x}$ is a regular sequence on M/M_r and $\underline{x}M \cap M_r = \underline{x}M_r$.

Friday, September 23. We began class by defining the nth symbolic power of a prime ideal P contained in the Noetherian ring R as follows: $P^{(n)} := P^n R_P \cap R$. One can readily see that $P^{(n)}$ is the P-primary component of P^n which in turn equals $\{r \in R \mid sr \in P^n, \text{for some } s \in R \backslash P\}$. It followed that $P^{(n)} = P^n$ if and only if $\text{Ass}(R/P^n) = \{P\}$. This in turn implies that there exists n_0 such that $P^{(n)} = P^n$ for all $n \geq n_0$ or $P^{(n)} \neq P^n$, for all $n \geq n_0$. We followed this with the example below.

Example. Let $R := \mathbb{Q}[x, y, z]/\langle x^2 - yz \rangle$ and $P := (x, y)R$. Then $P^{(2)} \neq P^2$. As an optional exercise, we asked whether or not $P^{(n)} = P^n$, for $n \geq 3$.

This was followed by a general discussion of the types problems related to symbolic powers that have been studied in the past and are currently being studied, including, when equality $P^{(n)} = P^n$ holds for all (or some) n, when the P-adic and P-symbolic topologies are equivalent, uniform equivalence of topologies and various containment problems, such as, given n, what it the least m such that $P^{(m)} \subseteq P^n$ (when such an m exists).

We then stated the Zariski-Nagata Theorem in the following form.

Zariski-Nagata Theorem. Set $R := k[x_1, \ldots, x_d]$, where k is an algebraically closed field of characteristic zero. If $P \subseteq R$ is a prime ideal, then $P^{(n)}$ consists of those $f \in R$ such that $f \in P$ and all mixed partials of order less than n belong to P.

We noted that since k is algebraically closed, if we let $X := V(P)$ denote the set of points $\underline{\alpha} \in k^d$ such that $f(\underline{\alpha}) = 0$, for all $f \in P$, then the Zariski-Nagata theorem can be interpreted as saying $P^{(n)}$ consists of all $f \in R$ such that f and all of its mixed partials of order less than n vanish on X.

We then defined $P^{\langle n \rangle}$ as the ideals consisting of those $f \in R$ such that $f \in P$ and all mixed partials of order less than n belong to P. Thus, the Zariski-Nagata theorem asserts that $P^{(n)} = P^{\langle n \rangle}$ for all n. We finished the class with the following, which gives half of the Zariski-Nagata theorem.

Proposition. For R and P as above, $P^{(n)} \subseteq P^{\langle n \rangle}$.

The proof of the proposition follows (more or less) immediately from the following observation: If f is a product of n elements from P, then any mixed partial derivative of f of order less n than belongs to P - using the product rule for derivative.

Wednesday, September 21. We began class by proving that if R is a Noetherian ring and $I \subseteq R$ is an ideal, then the sets $\text{Ass}(R/I^n)$ are eventually stable, if the sets $\text{Ass}(I^n/I^{n+1})$ are eventually stable. This was an immediate consequence of the behavior of associated primes in a short exact sequence, applied to the sequences

$$0 \to I/^n/I^{n+1} \to R/I^{n+1} \to R/I^n,$$

together with the fact that $\bigcup_{n \geq 1} \text{Ass}(R/I^n)$ is a finite set. We then had a brief discussion concerning *standard graded rings* of the form $R = \oplus_{n \geq 0} R_n$, with $R = R_0[R_1]$. This was followed by the statement of the following:

Proposition. Let R be a Noetherian standard graded ring.

(i) If $P \in \mathrm{Ass}_{R_0}(R_n)$ for some n, then there exists $Q \in \mathrm{Ass}_R(R)$ with $Q \cap R_0 = P$.
(ii) The sete of primes $\mathrm{Ass}_{R_0}(R_n)$ are stable for $n >> 0$.

Before proving the proposition, we noted that it immediately implies the stability of $\mathrm{Ass}(I^n/I^{n+1})$ - and hence $\mathrm{Ass}(R/I^n)$ - by applying part (ii) of the proposition to the standard graded ring $\mathcal{R}/t^{-1}\mathcal{R}$, where $\mathcal{R}$ denotes the extended Rees ring of R with respect to I. (Sorry: The R in this latter sentence is a general Noetherian ring, not the graded ring in the Proposition above.) For the proof of the proposition, we noted that (i) implies that $\bigcup_{R_0} \mathrm{Ass}_{R_0}(R_n)$ is finite. We then showed these sets are eventually increasing by proving that $(0 :_R R_1)_n = 0$ for $n >> 0$ in the standard graded ring R, where $(0 : R_1)_n$ denotes the set elements in R_n annihilating R_1 - which gives part (ii) of the proposition. The proof of this latter fact proceeded along the following lines: The ideal $(0 :_R R_1)$ can be generated by homogenous elements $f_1, \ldots, f_t$, where each $f_j \in R_{d_j}$. Taking $n_0 := 1 + \max\{d_j \mid 1 \leq j \leq t\}$, it followed that if $h \in (0 :_R R_1)_n$, with $n \geq n_0$, then we could write $h = a_1f_1 + \cdots + a_tf_t$, with each $a_j \in R_{n-d_j}$. Since R is a standard graded ring, and each $n - d_j > 0$, $a_j \in (R_1)^{n-d_j}$, so that each $a_jf_j = 0$, which showed $h = 0$.

Optional Exercises. Let R be a (standard) graded ring as discussed in class.

(i) Show that if R is Noetherian, then R_0 is Noetherian and each R_n is a finitely generated R_0-module.
(ii) Show that if R_0 is Noetherian and $R_+ := \oplus_{n>0} R_n$ is a finitely generated ideal, then R is Noetherian.
(iii) Show that an ideal $I \subseteq R$ can be generated by homogeneous elements if and only if $I = \oplus_{n\geq 0}(I \cap R_n)$. Such an ideal is called *homogeneous ideal* of R.
(iv) Let $Q \in \mathrm{Ass}_R(R)$. Show that Q is a homogeneous ideal and Q is the annihilator of a homogeneous element of R.

Monday, September 19. We began class by proving the classical theorem:

Theorem. Every ideal in a Noetherian ring has an irredundant primary decomposition.

For the proof of this theorem, we noted that it suffices to show that every ideal has a primary decomposition. The proof of this fact was almost the same as the proof from September 14 that cyclic modules have finitely many associated primes: If the statement is false, then there exists an ideal J maximal with respect to this failure. As before, there exist $a \in R$ and $n \geq 1$ such that $(J : a^n)$ and $\langle J, a^n\rangle$ properly contain J, with $J = (J : a^n) \cap \langle J, a^n\rangle$. A primary decomposition for the two larger ideals then gives rise to one for J, providing the necessary contradiction.

We then proved the following proposition:

Proposition. Let R be a Noetherian ring and $I \subseteq R$ an ideal. Let $q_1, \cdots q_r$ be those primary components of (0) in any primary decomposition of (0) satisfying $\sqrt{q_i} + I \neq R$. Then $\bigcap_{n\geq 1} I^n = q_1 \cap \cdots \cap q_r$.

This was followed by a discussion and some preliminary results that will lead to the interesting theorem of Brodmann that states if R is a Noetherian ring, and $I \subseteq R$ is an ideal, then there exists $n \geq n_0$ such that $\mathrm{Ass}(R/I^n) = \mathrm{Ass}(R/I^{n+1}) = \cdots$, for all $n \geq 1$.

Preliminary results. Let R be a Noetherian ring.

(i) Let S be Noetherian ring containing R and $J \subseteq S$ an ideal. If $P \in \mathrm{Ass}(R/(J \cap R))$, then there exists $Q \in \mathrm{Ass}(S/J)$ such that $Q \cap R = P$.
(ii) If $a \in R$ is a non-zerodivisor, then $\mathrm{Ass}(R/aR) = \mathrm{Ass}(R/a^nR)$, for all $n \geq 1$.
(iii) For any ideal $I \subseteq R$, $\bigcup_{n\geq 1} \mathrm{Ass}(R/I^n)$ is finite.

The proof of the crucial statement (iii) followed from (i) and (ii): If $P \in \mathrm{Ass}(R/I^n)$, for some n, then there exists $Q \in \mathrm{Ass}(\mathcal{R}/t^{-n}\mathcal{R})$ with $Q \cap R = P$, where $\mathcal{R}$ denotes the extended Rees algebra of R with respect to I (since $t^{-1}\mathcal{R} \cap R = I^n$, for all $n \geq 1$). By (ii) above, $\bigcup_{n\geq 1} \mathrm{Ass}(\mathcal{R}/t^{-n}\mathcal{R})$ is finite, which gives (iii).

Friday, September 16. We began class with the following important proposition.

Proposition. Let R be a Noetherian ring and M a finitely generated module over R. If an ideal $I \subseteq R$ consists of zerodivisors on M then there exists $0 \neq x \in M$ such that $I \cdot x = 0$.

The proof of this proposition followed from our early results in associated primes: The ideal I must be contained in the union of the associated primes of M and thus, contained in one such, say P (since M has just finitely many associated primes). But P is the annihilator of some x in M, so I annihilates x.

We followed the proposition with the following combination of definitions and consequences thereof:

Definition/Proposition. Let R be a Noetherian ring.

(i) An ideal $Q \subseteq R$ is *primary* if whenever $ab \in Q$ and $a \notin Q$, then $b \in \sqrt{Q}$.
(ii) If Q is primary, then $\sqrt{Q} = P$ is a prime ideal. We then say that Q is P-primary.
(iii) Given an ideal $I \subseteq R$, if $I = Q_1 \cap \cdots \cap Q_r$, with each Q_i primary, then $Q_1 \cap \cdots \cap Q_r$ is a *primary decomposition* of I.
(iv) The primary decomposition $I = Q_1 \cap \cdots \cap Q_r$ is *irredundant* if no Q_i can be deleted and have the intersection of the remaining ideals still equal I and $\sqrt{Q_i} \neq \sqrt{Q_j}$, if $i \neq j$.
(v) If I has a primary decomposition, it has an irredundant primary composition.
(vi) Q is P-primary if and only if there exists a prime ideal P such that $\mathrm{Ass}(R/Q) = \{P\}$.

Optional Exercise. Let R be a commutative ring and $I \subseteq R$ an ideal.

(i) Let $S \subseteq R$ be a multiplicative closed set. Then there exists a one-to-one correspondence between the primary ideals of R disjoint from S and the primary ideals of R_S.
(ii) Suppose I admits a primary decomposition and $S \subseteq R$ is a multiplicatively closed subset. Describe (with proof) the primary decomposition of I_S as an ideal of R_S.

We then proved the following crucial proposition:

Proposition. Let R be a Noetherian ring and $I \subseteq R$ an ideal. Suppose $I = Q_1 \cap \cdots \cap Q_r$ is a primary decomposition, with $\sqrt{Q_i} = P_i$. Then $\mathrm{Ass}(R/I) = \{P_1, \ldots, P_r\}$.

An immediate consequence of this proposition was that if $I = Q_1 \cap \cdots \cap Q_r = Q'_1 \cap \cdots \cap Q'_t$ are two irredundant primary decompositions of I, then $r = t$ and after re-indexing, $\sqrt{Q_i} = \sqrt{Q'_i}$, for $1 \leq i \leq r$. Thus, irredundant primary decompositions are unique only up to the number of terms, and the radicals of the primary components appearing in the decomposition. We ended class with the following discussion which shows that an ideal may have infinitely many irredundant primary decompositions.

Discussion. Suppose R is a Noetherian ring and I an ideal such that not every prime in $\mathrm{Ass}(R/I)$ is a minimal prime over I. Let
$$I = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_n$$
be an irredundant primary decomposition with $\sqrt{Q_i} = P_i$ for all $1 \leq i \leq n$ and such that $P_1, \ldots, P_r$ are the primes minimal over I. (Note: The primes $P_{r+1}, \ldots, P_n$ are called the *embedded prime divisors of I*.) Choose $m_i \geq 1$ such that $P_i^{m_i} \subseteq Q_i$ and set $Q_i(m_i) := (I + P_i^{m_i})_{P_i} \cap R$, for $r+1 \leq i \leq n$. Then each $Q_i(m_i)$ is also P_i primary and
$$I = Q_1 \cap \cdots \cap Q_t \cap Q_{r+1}(m_{r+1}) \cap \cdots \cap Q_n(m_n)$$
is an irredundant primary decomposition. Letting the m_i increase shows that I has infinitely many irredundant primary decompositions.

Optional Exercise. Suppose
$$I = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_n = Q'_1 \cap \cdots \cap Q'_r \cap Q'_{r+1} \cap \cdots \cap Q'_n$$
are primary decompositions with $\sqrt{Q_i} = \sqrt{Q'_i} = P_i$ for all $1 \leq i \leq n$ and such that $P_1, \ldots, P_r$ are the primes minimal over I. Prove that $Q_i = Q'_i$, for $1 \leq i \leq r$.

Wednesday, September 14. Today we began the the third part of the course which is devoted to prime divisors and symbolic powers of prime ideals. We began with the following definition:

Definition. Let R be a Noetherian ring and M an R-module (not necessarily finitely generated). A prime ideal $P \subseteq R$ is an *associated prime* of M or a *prime divisor* of M if $P = \mathrm{ann}(x) := \{r \in R \mid rx = 0\}$. We will also write $P = (0 :_R x)$ or just $P = (0 : x)$ if there is no ambiguity about the ring. We write $\mathrm{Ass}(M)$ for the set of associated primes of M. We then offered the following (optional) exercise:

Exercise. Using the definition above, show that $P \in \mathrm{Ass}(M)$ if and only if $P_P \in \mathrm{Ass}_{R_P}(M_P)$.

This was followed by the statement and proof of the following (standard) proposition:

Proposition. Let R be a Noetherian ring and M an R-module.

(i) $M \neq 0$ if and only if $\text{Ass}(M) \neq \emptyset$.
(ii) $\bigcup\{P \mid P \in \text{Ass}(M)\}$ is the set of zerodivisors on M.
(iii) If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then $\text{Ass}(B) \subseteq \text{Ass}(A) \cup \text{Ass}(C)$.
(iv) If M is a finitely generated R-module, then $\text{Ass}(M)$ is a finite set.

Some salient comments about the proof: The proof of (i) showed that the annihilator of any non-zero element of M is contained in an associated prime, by showing that for any $0 \neq x \in M$, an ideal maximal among proper ideals of the form $\text{ann}(rx)$ is prime. This immediately gives (ii). The proof of (iii) was straightforward. The proof of (iv) reduced to the case that M is isomorphic to R/I for $I \subseteq R$ an ideal. If the statement in (iv) were false, there would exist $J \subseteq R$ an ideal maximal with respect to the property that R/J has infinitely many associated primes. We then showed there exists $a \in R$ such that for n sufficiently large, $(J : a^n)$ and $\langle J, a^n\rangle$ properly contain J and their intersection equals J. From there it was easy to see that an associated prime of R/J was either an associated prime of $R/(J : a^n)$ or an associated prime of $R/\langle J, a^n\rangle$. Maximality of J implies the latter modules have finitely many associated primes, and thus R/J has finitely many associated primes, providing the necessary contradiction.

We ended class by first showing that if $J \subseteq R$ is an ideal and P is a prime minimal over J, then $P \in \text{Ass}(R/J)$. We then definied the concept of a *primary* ideal. *The ideal $Q \subseteq R$ is primary if whenever $ab \in Q$ and $a \notin Q$, then $b^n \in Q$, for some n.*

Monday, September 12. The goal of today's lecture was to prove the following theorem.

Main Theorem for Part 2 of the course. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n-variables over a field k having characteristic zero. If $P \subseteq R$ is a prime ideal, then $P = \bigcap_{M\in\mathcal{C}} M$, where $\mathcal{C}$ denotes the set of maximal ideal M containing P such that $(R/P)_M$ is a regular local ring.

We began with two preliminary results. The first result was the standard result that if B is an integrally closed domain with quotient field L, and ω is integral over B, for ω belonging to an integral domain containing B, then the minimal polynomial $f(W)$ for ω over L belongs to $B[W]$ and $B[\omega] = B[W]/\langle f(W)\rangle$. The second preliminary result was to show that if B is a locally regular Noetherian integral domain, then B is integrally closed. The proof of this latter result used the following (optional) exercise:

Exercise. Let R be a Noetherian ring, $b \in R$ a non-zerodivisor and $a \in R$ is such that $(bR : a)$ is a proper ideal. Prove that if the prime ideal P is minimal over $(bR : a)$, then there exists $c \in R$ such that $P = (bR : c)$. Conclude that if R is an integral domain, then $R = \bigcap_{P\in\mathcal{P}} R_P$, where $\mathcal{P}$ is the set of prime ideals in R that can be written as $P = (bR : c)$, for some non-zero $b, c \in R$.

After these preliminary results, we presented the following theorem, which was key to our main result.

Theorem. Suppose A is an integral domain that is a finitely generated k-algebra, where k is a field of characteristic zero. Then there exists $\gamma \in A$ such that A_γ is locally regular.

The idea of the proof of this theorem was the following: By the Noether Normalization theorem, there exists a polynomial ring B in $\dim(A)$ variables over k such that A is a finite extension of B. The characteristic zero assumption guarantees the existence of a primitive element ω for the corresponding extension of quotient fields, and one may assume $\omega \in A$. If one shows that there exists $\gamma \in B[\omega]$ such that $B[\omega]_\gamma$ is locally regular, then since $B[\omega]_\gamma$ is integrally closed, $B[\omega]_\gamma = A_\gamma$, so A_γ is locally regular. The proof then showed that $\gamma := f'(\omega)$ works, where $f(W)$ is the minimal polynomial for ω over the quotient field of B.

For the proof of the main theorem, we noted that it was enough to show that if A is an integral domain that is a finitely generated k-algebra, where k is a field of characteristic zero, then $(0) = \bigcap_{M\in\mathcal{C}} A_M$ where $\mathcal{C}$ denotes the maximal ideals M in A such that A_M is a regular local ring. Take $\gamma \in A$ as in the previous theorem. Then A_γ is a Hilbert ring and thus, (0) in this ring is the intersection of the maximal ideals of A_γ. But each maximal ideal in A_γ is of the form M_γ where $M \subseteq A$ is a maximal ideal not containing γ.

Moreover, A_M is a regular local ring for each M not containing γ thus,

$$\bigcap_{M\in\mathcal{C}} A_M \subseteq \bigcap_{\gamma\notin M} A_M = \bigcap_{\text{max ideals}} (A_\gamma)_{M_\gamma} = (0),$$

which gives the result.

During class, we also discussed the geometric interpretation of the main theorem above. Suppose that, $R = k[x_1, \ldots, x_n]$ is the polynomial ring in n-variables over an algebraically closed field k having characteristic zero. Let $P \subseteq R$ be a prime ideal and set $X := V(P)$, an irreducible variety in k^n. Thus, each maximal ideal M containing P corresponds to a point on X. If $A := R/P$, the coordinate ring of X, then the main theorem shows that A is the intersection of A_M where the intersection is taken over the maximal ideals of A corresponding to the nonsingular points of X. Moreover, the proof of the theorem stated after the exercise shows that the set of non-singular points of X contains a Zariski open subset of X

Friday, September 9. We began class by reviewing the second version of the weak Nullstellensatz given in the previous lecture and defining the ingredients of the full Nullstellensatz:

Hilbert's Nullstellensatz. Let k be an algebraically closed field, $J \subseteq R := k[x_1, \ldots, x_n]$ an ideal and $V(J)$ the set of points $\underline{\alpha} \in k^n$ such that $f(\underline{\alpha}) = 0$, for all $f \in J$. If $I(V(J))$ denotes the set of $g \in R$ such that $g(\underline{\beta}) = 0$, for all $\underline{\beta} \in V(J)$, then $I(V(J)) = \sqrt{J}$.

The proof of the theorem was then an immediate consequence of the definitions, the weak form of the theorem, the fact that the polynomial ring in question is a Hilbert ring and the fact that $f(x_1, \ldots, x_n)$ vanishes at $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ if and only if $f(x_1, \ldots, x_n) \in M_{\underline{\alpha}}$, where $M_{\underline{\alpha}} = \langle x_1 - \alpha_1, \ldots, x_n - \alpha_n\rangle$.

We then proved the Noether normalization lemma in the following form:

Noether Normalization. Let A be an integral domain of dimension d which is a finitely generated k algebra, for k an infinite field. Then there exist $x_1, \ldots, x_d \in A$ such that $x_1, \ldots, x_d$ are algebraically independent over k and A is a finite module extension of $B := k[x_1, \ldots, x_d]$.

The proof of Noether normalization was by induction on n, where $A := k[u_1, \ldots, u_n]$. This was achieved by observing that if $f(z_1, \ldots, z_n)$ is a polynomial in n variables over k, we may change to new variables $y_1, \ldots, y_n$ so that $f(y_1, \ldots, y_n)$ is monic in y_n (up to a unit multiple). The following fact, left as an (optional) exercise, was crucial:

Exercise. Let $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial, with k an infinite field. Prove there exist $a_1, \ldots, a_{n-1} \in k$ such that $f(a_1, \ldots, a_{n-1}, 1) \neq 0$.

Wednesday, September 7. We began class by reviewing the definitions of G-domain, G-ideal and Hilbert ring and recalling the properties of a Hilbert ring R:

(i) R/J is a Hilbert ring for all ideals $J \subseteq R$.
(ii) R is a Hilbert ring and a G-domain if and only if R is a field.
(iii) If $M \subseteq R[x]$ is a maximal ideal, then $M \cap R$ is a maximal ideal.

This was followed by showing that if R is a Hilbert ring, and $f \in R$ is not nilpotent, then R_f is a Hilbert ring and any maximal ideal $Q \subseteq R_f$ is of the form M_f, where $M \subseteq R$ is a maximal ideal. With these results in hand we were able to prove:

Theorem. If R is a Hilbert ring, then $R[x]$, the polynomial ring in one variable over R, is a Hilbert ring.

An immediate consequence of this theorem is that the polynomial rings $k[x_1, \ldots, x_m]$ and $\mathbb{Z}[x_1, \ldots, x_n]$ are Hilbert rings, where k is a field and $\mathbb{Z}$ denotes the ring of integers.

Optional Exercises. Suppose k is a field and $R = k[x_1, \ldots, x_n]$.

(i) Prove that if $M \subseteq R$ is a maximal ideal, then there exist $f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n) \in R$ such that $M = \langle f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n)\rangle$.
(ii) Suppose $f(x_1, \ldots, x_n) \in R$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in k^n$. Show that there exist $g_1, \ldots, g_n \in R$ such that $f(x_1, \ldots, x_n) = g_1 \cdot (x_1 - \alpha_1) + \cdots + g_m \cdot (x_n - \alpha_n)$.

We then proved the weak Nullstellensatz in the following two forms.

Weak Nullstellensatz. Let k be a field and $R := k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. Suppose $M \subseteq R$ is a maximal ideal. Then:

(i) R/M is an algebraic extension of k.
(ii) If k is algebraically closed, then $M = \langle x_1 - \alpha_1, \ldots, x_n - \alpha_n \rangle$ for $\alpha_i \in k$.

The proof of (ii) followed readily from (i), while (i) followed by induction on n, and the facts that $S := k[x_1, \ldots, x_{n-1}]$ is Hilbert and $M \cap S$ is a maximal ideal. We ended class by discussing the ingredients of the full form of the Nullstellensatz and noted that it too follows readily from the definitions and the fact that $k[x_1, \ldots, x_n]$ is a Hilbert ring. The details will be given in Friday's lecture.

Friday, September 2. We began the second part of the course which is devoted to Hilbert rings and applications. The motivation for this section is as follows: One of our main goals in the course is the Zariski-Nagata theorem, and for this we need the fact that if R is a polynomial ring in several variables over a field of characteristic zero, then any non-zero prime $P \subseteq R$ is the intersection of the maximal ideals $M \subseteq R$ such that $P \subseteq M$ and $(R/P)_M$ is a regular local ring. A first step towards this latter result is to show that any polynomial ring in finitely many variables over a field has the property that any prime ideal is the intersection of the maximal ideals containing it, i.e, such a polynomial ring is a *Hilbert ring*.

Unless noted otherwise, in this part of the course, R is an integral domain with quotient field K. We then gave the following definition: R is a G-domain if there exists $0 \neq a \in R$ such that $R_a = K$, where R_a denotes R localized at the multiplicatively closed set $\{1, a, a^2, \ldots\}$. We then showed that the following conditions are equivalent:

(i) R is a G-domain
(ii) K is a finitely generated R-algebra
(iii) There exists $0 \neq a \in R$ such that a belongs to every non-zero prime ideal of R.
(iv) There exists a maximal ideal $M \subseteq R[x]$, the polynomial ring in one variable over R, with $(0) = M \cap R$.

We then offered the following (optional):

Exercise. Let $R \subseteq S$ be integral domains and $0 \neq s \in S$. If $R[s]$ is a G-domain, then s is algebraic over R and R is a G-domain.

We followed this by showing that if R is a Noetherian domain, then R is a G-domain if and only if R has Krull dimension one and just finitely many maximal ideals. This was an immediate consequence of the definition, the fact that any ideal in a Noetherian ring has only finitely many minimal primes ideals, and the consequence of Krull's principal ideal theorem proven at the end of class on Wednesday August 24.

We then defined a *G-ideal* to be a prime ideal $P \subseteq R$ such that R/P is a G-domain. Here we do not require R to be an integral domain. Thus, $P \subseteq R$ is a G-ideal if and only if there exists a maximal ideal $M \subseteq R[x]$ with $M \cap R = P$. We then discussed how the standard proof that the radical of an ideal in a commutative ring is the intersection of the prime ideals containing the ideal really shows that the radical of every ideal is an intersection of G-ideals

We then showed the following:

Proposition/Definition. Let R be a commutative ring. The following are equivalent:

(i) Every G-ideal is a maximal ideal
(ii) Every prime ideal is an intersection of maximal ideals

A commutative ring satisfying these conditions is called a *Hilbert ring*. The proof of the proposition made use of the observation: A commutative ring is a G-domain and a Hilbert ring if and only if it is a field.

And we ended class by noting that it follows from what we have done that if R is a Hilbert ring and $M \subseteq R[x]$ is a maximal ideal, then $M \cap R$ is a maximal ideal.

Wednesday, August 31. The purpose of today's lecture was to prove the following theorem:

Theorem. Let $(R, \mathfrak{m})$ be a regular local ring and $P \subseteq R$ a non-zero prime ideal. If R/P is a regular local ring, then any minimal generating set for P extends to a minimal generating set of $\mathfrak{m}$. In particular, a minimal generating set for P forms a regular sequence.

To facilitate the proof of this theorem, we first showed that if the ideal $J = \langle y_1, \ldots, y_n \rangle$ is generated by a regular sequence, then the kernel K of the map $\phi : R^n \to J \to 0$ given by $\phi \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1y_1 + \cdots + a_ny_n$ is generated by the $\binom{n}{2}$ Koszul relations:

$$\begin{pmatrix} -y_2 \\ y_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -y_3 \\ 0 \\ y_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -y_n \\ y_1 \end{pmatrix}, \begin{pmatrix} 0 \\ -y_3 \\ y_2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -y_n \\ y_{n-1} \end{pmatrix}.$$

As a corollary we were able to deduce that if $I, J \subseteq R$ are ideals and $J = \langle y_1, \ldots, y_n \rangle$ is generated by elements forming a regular sequence modulo I, then $I \cap J = IJ$. We then proceeded with the proof of the theorem.

The idea of the proof is as follows: If $J = \langle y_1, \ldots, y_n \rangle$ is generated by elements forming a minimal generating set for $\mathfrak{m}/P$, then we have $P \cap J = PJ$. We then showed that the elements $y_1, \ldots, y_n$ form part of a minimal generating set for $\mathfrak{m}$. Thus, R/J is a regular local ring in which $\mathfrak{m}/J = (P + J)/J$. Therefore, there exists $\underline{x} = x_1, \ldots, x_t \in P$ such that their images in R/J form a minimal generating set for $\mathfrak{m}/J$. It followed that $\{\underline{x}, \underline{y}\}$ forms a minimal generating set for $\mathfrak{m}$ and $\underline{x}$ is a regular sequence. Nakayama's lemma and the relation $P \cap J = PJ$ yields $P = \langle \underline{x} \rangle$, a minimal generating set of P. Finally, if $z_1, \ldots, z_t$ is a minimal generating set for P, $\underline{z}, \underline{y}$ is easily seen to be a minimal generating set for $\mathfrak{m}$.

Monday, August 29. We began class by noting that the proof of the theorem $\dim(R[x]) = \dim(R) + 1$ for a Noetherian ring R from the previous lecture can be modified to show that if R is a locally regular Noetherian ring, then the polynomial ring $R[x]$ is also locally regular. As a corollary we noted that if k is a field, then the polynomial rings $k[x_1, \ldots, x_n]$ and $\mathbb{Z}[x_1, \ldots, x_n]$ are locally regular. We then proved the following proposition concerning systems of parameters and minimal generating sets for the maximal ideal in a regular local ring:

Proposition. Let $(R, \mathfrak{m})$ be a local ring of Krull dimension d.

(i) If $\underline{x} = x_1, \ldots, x_d$ is a system of parameters, then $\dim(R/\langle x_1, \ldots, x_i \rangle) = d - i$, for all $1 \leq i \leq d$.
(i) If R is a regular local ring and $\mathfrak{m} = \langle \underline{x} \rangle$ as in (i), then $R/\langle x_1, \ldots, x_i \rangle$ is a regular local ring.
(iii) If R is a regular local ring and $x \in \mathfrak{m}$, then $R/\langle x \rangle$ is a regular local ring if and only if $x \in \mathfrak{m} \backslash \mathfrak{m}^2$.

We followed this proposition by showing that a regular local ring is an integral domain. This, combined with the previous proposition, shows that the maximal ideal in a regular local ring is generated by a *regular sequence*.

Definition. Let R be a Noetherian ring and M a finitely generated R-module. A sequence of elements $\underline{x} = x_1, \ldots, x_d$ in R is a *regular sequence on M* if:

(i) $\langle \underline{x} \rangle M \neq M$
(ii) x_1 is not a zero divisor on M and for $i > 1$, x_{i+1} is not a zero divisor on $M/\langle x_1, \ldots, x_i \rangle M$.

We ended class by mentioning the following result, to be shown in the next lecture: If $(R, \mathfrak{m})$ is a regular local ring and $P \subseteq R$ is a prime ideal such that R/P is also a regular local ring, then a minimal generating set for P can be extended to a minimal generating set for $\mathfrak{m}$. In particular, P is generated by a regular sequence.

The following (optional) exercises are related to today's lecture.

Exercise. (i) Let R be a commutative ring. Prove that in the polynomial ring $R[x]$ there does not exist a chain of primes $Q_1 \subsetneq Q_2 \subsetneq Q_3$ such that $Q_1 \cap R = Q_2 \cap R = Q_3 \cap R$. Now suppose R is Noetherian and $P \subseteq R$ is a prime ideal. Prove that $\text{height}(P) = \text{height}(P[x])$.

Exercise. Let $(R, \mathfrak{m})$ be a local ring and $0 \neq x \in \mathfrak{m}$. Prove that $\mu(\mathfrak{m}) = \mu(\mathfrak{m}/\langle x \rangle)$ if and only if $x \in \mathfrak{m}^2$.

Friday, August 26. We began class by proving Krull's height theorem as stated in the synopsis of the previous lecture. An immediate consequence of this theorem is that if $(R, \mathfrak{m})$ is a local ring, whose maximal ideal is generated by n elements, then $\dim(R) \leq n$. Thus, $\dim(R)$ is a lower bound for the number of generators of the maximal ideal in a local ring R. We then defined a *regular local ring*, abbreviated RLR, to be a local ring whose maximal ideal can be generated by $\dim(R)$ elements. We then stated the following (optional) exercise:

Exercise. M be a finitely generated module over the local ring $(R, \mathfrak{m})$. Then $x_1, \ldots, x_n \in M$ are a minimal set of generators for M if and only if their images in $M/\mathfrak{m}M$ form a basis for the vector space $M/\mathfrak{m}M$ over $k := R/\mathfrak{m}$.

It follows from the exercise that any two minimal generating sets for M have the same number of elements. We denote this common number by $\mu(M)$. We also noted that if I is an ideal in the local ring $(R, \mathfrak{m})$ such that $\sqrt{I} = \mathfrak{m}$ and $d = \dim(R)$, then $\mu(I) \geq d$. When $\mu(I) = d$, we noted that the generators for I are a *system of parameters* for R. We then proved a proposition stating that if P is a prime ideal in a Noetherian ring such that $\text{height}(P) = c$, then there exist $x_1, \ldots, x_c \in P$ such that P is minimal over $J := \langle x_1, \ldots, x_c \rangle$. In addition, $\text{height}(J) = c$ and $\text{height}(Q) = c$, for every minimal prime Q of J.

We followed the previous proposition by stating the following (optional) exercise.

Exercise. Let R be a Noetherian ring. Prove that R satisfies the descending chain condition on primes ideals. In particular, show that every ideal has finite height.

We ended class by proving that if R is a Noetherian ring of Krull dimension d, then the polynomial ring $R[x]$ Krull has dimension $d + 1$. The proofs of this theorem and the previous proposition used Krull's height theorem.

Wednesday, August 24. We discussed how our next immediate goal is to prove Krull's height theorem: *Let R be a Noetherian ring and I an ideal generated by n elements. If P is a prime minimal over I, then the height of P is less than or equal to n.* We then proved the tangentially related result that any ideal in a Noetherian ring has just finitely many prime ideals minimal over it. We then defined the notion of length for an R-module M: The module M has *finite length* if it has a composition series, i.e., there exists a sequence of submodules $(0) = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$, where each M_{i+1}/M_i is a simple module. It follows from the Jordan-Hölder theorem that every composition series has the same number of terms, which we defined to be the length of M, denoted $\lambda(M)$. We then mentioned two (optional) exercises:

Exercise. Show that an R-module has finite length if and only if it is both Artinian and Noetherian.

Exercise. If $0 \to A \to B \to C \to 0$ is a short exact sequence of R modules, prove that B has finite length if and only if A nd C have finite length, in which case, $\lambda(B) = \lambda(A) + \lambda(C)$.

We then showed that a Noetherian ring with one prime ideal is Artinian, which is a special case of the fact that a commutative ring R is Artinian if and only if R is Noetherian and zero dimensional. With this in hand we were able to prove:

Krull's Principal Ideal Theorem. Let R be a Noetherian ring and aR a non-zero principal ideal. If P is a prime minimal over aR, then the height of P is less than or equal to one.

The proof we gave of the principal ideal theorem was the one given by Rees in the 1955 paper referred to in the previous lecture. We finished the lecture with the following interesting application of the principal ideal theorem: Let R be a Noetherian ring and $P' \subsetneq P$ prime ideals. If there exists a prime ideal Q with $P' \subsetneq Q \subsetneq P$, then there are infinitely many such Q.

Monday, August 22. We began class with a brief overview of the topics to be covered this semester. First up: some preliminary material, starting with the following version of Krull's intersection theorem:

Krull's Intersection Theorem. Let R be a Noetherian ring, $I \subseteq R$ an ideal and $J := \bigcap_{n \geq 1} I^n$. Then $x \in J$ if and only if there exists $a \in I$ such that $(1 + a)x = 0$.

The proof we gave is the one given by D. Rees in his celebrated 1955 paper *Two classical theorems of ideal theory.* In this paper Rees introduced what is now called the *extended* Rees algebra of R with respect to I, namely $\mathcal{R} := R[It, t^{-1}]$, t an indeterminate, which is a subring of the Laurent polynomial ring $R[t, t^{-1}]$. It is easy to check that $t^{-n}\mathcal{R} \cap R = I^n$ for all n, which enabled Rees to reduce the general case of the theorem to the case where I is generated by a single non-zerodivisor. The following statements follow immediately from the theorem:

(i) There exists $a_0 \in I$ such that $(1 + a_0) \cdot J = 0$.
(ii) $J = 0$ if R is an integral domain or I is contained in the Jacobson radical of R. In particular $J = 0$ for every proper ideal I in a local ring.

The same paper of Rees introduced the Artin-Rees Lemma in the following form: If R is a Noetherian ring and I, J are ideals, then there exists $k \geq 1$ such that $I^n \cap J = I^{n-k}(I^k \cap J)$ for all $n \geq k$. (Note: This is not one of the two theorems in the title of Rees's paper.) We gave Rees's proof of this result, again using the extended Rees algebra. We ended class with the following corollary of the Artin-Rees Lemma: Let R be a Noetherian ring, $I \subseteq R$ an ideal and $0 \neq x \in R$. Then there exists k such that $(I^n : x) = (0 : x) + I^{n-k}(I^k : x)$, for all $n \geq k$.

The following (optional) exercises was mentioned in class:

Exercise. Formulate and prove versions of the Krull intersection theorem and the Artin-Rees lemma for finitely generated modules over a Noetherian rings.